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Regularity of Minimizers of One-Dimensional Scalar Variational Problems with Lagrangians with Reduced Smoothness Conditions

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1 Introduction

Consider the following general problem of one-dimensional variational calculus. Let $\mathscr{F} : \mathcal{C} \to \mathbb{R}$ be an integral functional

$$\mathscr{F}(u) = \int_I F(x, u(x), u'(x))\,\mathrm{d}x$$

where I is some interval and $\mathcal{C}$ is some domain of functions for which the Lagrangian $F(x, z, p)$ and the integral is well defined. We will consider functionals defined on a domain of functions with fixed boundary conditions i.e. $\forall u \in \mathcal{C}$ we have $u = u_0$ on ∂I for a certain fixed $u_0 \in \mathcal{C}$. We wish to find the lower bound for the values of $\mathscr{F}$ and if possible the minimizers in $\mathcal{C}$. The functions of $\mathcal{C}$ are called competitors.

Classical techniques in the calculus of variations were the so called indirect methods which consist in finding necessary conditions for extremals to variational problems. However even for fairly simple functionals the existence of minimizers among regular functions fails and the classical approach couldn't give a satisfactory answer or criteria on when this happens.

These problems led to the development of the so called direct methods, one of the main contributors of which was Tonelli. These methods attack minimizing problems for functional by suitably expanding the class of competing functions to guarantee the existence of minimizers. The most natural spaces on which to consider these problems are the Sobolev spaces, which in particular correspond to the class of AC functions for problems in one-space.

While Tonelli believed that AC was the right space to attack one-dimensional variational problems, results due to Lavrentiev [7] and others have shown that pathological behavior can arise when extending functionals to the class AC, and difficulty was encountered when trying to merge the new results with classical indirect methods.

There has been a need to study the criteria for when AC minimizers possess additional regularity or coincide with extremals in the classical sense. One of the most important advances in this direction was made again by Tonelli in the case of one-dimensional scalar problems in what is known as Tonelli's Partial Regularity Theorem. While multi-dimensional problems are of significantly different nature and Tonelli's Partial Regularity Theorem is not applicable there, many results from the scalar case extend to the vector-valued case. However, we will concentrate only on one-dimensional problems in scalar settings.

First let us define what we mean by Tonelli-like regularity. We will use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. The topology on $\overline{\mathbb{R}}$ is the compactification of the topology generated by the inclusion of $\mathbb{R}$. More precisely the topology on $\overline{\mathbb{R}}$ is generated by the open intervals (a, b), by $(b, +\infty]$ and $[-\infty, a)$ with $a, b \in \mathbb{R}$.

Definition 1.1. *We define the class of continuous extended-value functions $C(I; \overline{\mathbb{R}})$ for an interval I to be the functions $f : I \to \mathbb{R} \cup \{\pm\infty\}$ that are continuous on $I \setminus f^{-1}(\{\pm\infty\})$ and if $f(x_0) = +\infty(-\infty)$ then $\lim_{x \to x_0} f(x) = +\infty(-\infty)$.*

Notice that this definition coincides with the notion of continuous functions from I to $\overline{\mathbb{R}}$ considered with its topology.

Definition 1.2. *Consider a function $u \in AC(I)$ for some interval I. We say that u is regular in the sense of Tonelli (or has partial regularity) if the classical derivative $u'(x_0) := \lim_{x \to x_0} \frac{u(x)-u(x_0)}{x-x_0}$ exists in every point with possible values in $\mathbb{R} \cup \{\pm\infty\}$. Furthermore we require $u' \in C(I; \overline{\mathbb{R}})$. We call $E = (u')^{-1}(\{\pm\infty\})$ the singular set.*

Notice that E is closed by continuity, and since $u \in AC$, E has zero Lebesgue measure.

As stated above many results are extendable to vector-valued functions. However these generalizations are often of technical nature and usually are slightly weaker since the standard compactification of $\mathbb{R}^d$ is different from $\overline{\mathbb{R}}$ for $d = 1$, as no distinction between $+\infty$ and $-\infty$ is made.

The original formulation of Tonelli's Regularity Theorem applies to minimizers of functionals with smooth Lagrangians strictly convex in p. The present thesis is centered around the result of Tonelli and is meant to be a survey of the original formulation and more recent developments around this result. In particular we discuss the optimality and the necessity of all the hypotheses.

In the first two sections of this thesis we will provide a proof due to Ball and Mizel [1] of the original result that is applicable to functionals with C^3 Lagrangians. Next we provide the results of Mania [8] which show that the singular set is in general not empty and of Davie [4] that proves that Tonelli's description of the singular set is optimal. In particular any closed set of zero Lebesgue measure can occur as a singular set of a minimizer.

There has been much research and many results investigating necessary and sufficient conditions on Lagrangians for Tonelli-like regularity of minimizers. In most cases strict convexity remains central to proving these regularity results. Many of these results deal with conditions needed to move to full regularity of minimizers. For an example we refer to Clarke and Vinter [3]. However Tonelli's regularity result is noteworthy in that it requires no conditions on the Lagrangian other than smoothness and convexity. An important detail is that growth conditions are not mentioned in the theorem. In particular growth and coercivity conditions are necessary for most existence results, however, if the existence of the minimizer is provided by some other means, Tonelli's Partial Regularity Theorem allows one to obtain partial regularity. Most developments of partial regularity results manage to lower the smoothness hypothesis, however they do this requiring the Lagrangian to have some specific form or by assuming some kind of growth conditions on the Lagrangian. This thesis will not concentrate on these results since such additional hypotheses stray away from the original spirit of Tonelli's Partial Regularity Theorem.

Very recent results of Ferriero [5] answered the question of lowering the regularity hypothesis of Tonelli's partial regularity theorem without adding growth conditions and proving the result directly, using only convexity. Ferriero proves Tonelli's theorem for continuous Lagrangians that are Lipschitz continuous in z locally uniformly in z and p and with a Lipschitz constant integrable in x. The third section is dedicated to this result.

The fourth section is dedicated to the results of Gratwick and Preiss [6], published one year prior to Ferriero's paper. They construct counterexamples to Tonelli's Partial Regularity Theorem in continuous settings. In particular they prove that Tonelli-like regularity fails if sole continuity in the three variables is required.

Considering these two results we deem noteworthy that to obtain partial regularity of minimizers the only additional hypothesis other than convexity in p and continuity of the Lagrangian is a Lipschitz condition on z. In particular nothing other than continuity is required in x, while convexity accounts for the needed regularity in p.

2 Tonelli's partial regularity theorem

Theorem 2.1 (Tonelli's partial regularity theorem). *Let $\mathscr{F}$ be a functional with a Lagrangian $F(x,z,p)$ of class C^3 that satisfies $F_{pp}(x,z,p) > 0$ for all $(x,z,p) \in U \times \mathbb{R}$ where $U \subset \overline{I} \times \mathbb{R}$ is an open neighborhood of the graph of a certain $w \in AC(I,\mathbb{R})$. Suppose that w is a local (in AC or $W^{1,1}$ norm) minimizer for $\mathscr{F}$ in the class $\mathcal{C} = \{u \in AC(I,\mathbb{R}),\ u = w \text{ on } \partial I\}$. Then w has a classical derivative $w' \in C^0(\overline{I}, \mathbb{R} \cup \{\pm\infty\})$. The singular set $E = \Big\{x \in \overline{I} \mid w'(x) \in \{\pm\infty\}\Big\}$ is closed and has zero Lebesgue measure. $w \in C^3(\overline{I} \setminus E)$ and if the Lagrangian is C^k with $k \geq 3$ then w is $C^k(\overline{I} \setminus E)$.*

To prove this theorem we first study the classical solutions the Euler equation (2.1) for $\mathscr{F}$, which by definition are at least C^2. Then we prove that on a sufficiently small interval around a point where w' exists and is finite, there exists a local classical solution to Euler's equation with the same boundary values as w. By adapting classical results from indirect methods to functions of class AC we will show using convexity, that on the small interval a classical extremal strictly minimizes the functional even among AC with fixed boundary values. By minimality, w must coincide with the minimizing local extremal thus must be regular.

Then we study the behavior of w near the points of the singular set where the classical derivative w' is infinite or not defined. This will be done by a similar technique as above, where we again compare w with local solutions to Euler's equation.

2.1 Some notes about solutions of Euler's equation

Next is a lemma about the regularity of solutions to the Euler equation and the regularity of the dependence on the initial conditions. Even though we will initially apply this lemma during the proof of theorem 2.1 and there is a simpler proof of the required regularity of the dependence on the initial conditions in case of a Lagrangian of class C^k with $k \geq 3$ we prefer to give a more detailed analysis of the solutions of Euler's equation that we will find useful in subsequent results in less regular cases.

Let us first define what we mean by regularity of the solutions to a differential equation.

Definition 2.2. *Consider an ODE possibly in implicit form as (2.1). We say that the solutions to the ODE are of class C^k and have C^k dependence on initial data in an open set $A \subset \overline{I} \times \mathbb{R} \times \mathbb{R}$ if there exists a C^k mapping*

$$\begin{array}{rcrcl} u & : & A & \longrightarrow & \mathbb{R} \\ & & (x,a,b) & \longmapsto & u(x;\, a,b) \end{array}$$

such that for some fixed $x_0 \in \overline{I}$

$$\begin{aligned} u(x_0;\, a, b) &= a \\ \frac{\mathrm{d}}{\mathrm{d}x} u(x_0;\, a, b) &= b \\ u(x;\, a, b) \text{ satisfies the ODE in } x &\quad \forall (x, a, b) \in A \end{aligned}$$

and the solution with given initial conditions is unique i.e. if $v(x_0) = a$ *and* $v'(x_0) = b$ *then* $v(x) = u(x;\, a, b)$ *where* u *is defined.*

Lemma 2.3 (Regularity properties of solutions of Euler's equation)**.** *Let* $F(x, z, p)$ *be a* C^k *Lagrangian with* $k \geq 2$ *defined on* $\overline{I} \times \mathbb{R} \times \mathbb{R}$ *with* $F_{pp} > 0$*. Let* $A \subset \overline{I} \times \mathbb{R}$ *be a bounded open set. Then for any choice of* $N, M, \delta, \widetilde{\delta} > 0$ *one can find* $\varepsilon > 0$ *so that* $\forall (x_0, u_o) \in A$ *and for* $\|a\| < N$*,*$\|b\| < M$ *there is a unique* C^k *solution* $u(x;\, a, b)$ *of the Euler's equation:*

$$\frac{\mathrm{d}}{\mathrm{d}x} F_p(x, u, u') = F_z(t, u, u') \tag{2.1}$$

for $x \in (x_0 - \varepsilon; x_0 + \varepsilon) \cap \overline{I}$ *satisfying the initial conditions*

$$\begin{aligned} u(x_0;\, a, b) &= u_0 + a \\ u'(x_0;\, a, b) &= b \end{aligned}$$

The dependence on initial data of the solutions is of class C^{k-2}*. Furthermore, if* $k \geq 3$ *the following estimates*

$$\begin{aligned} |u'(x;\, a, b) - b| &< \delta \\ |\partial_a u(x;\, a, b) - 1| &< \widetilde{\delta} \end{aligned} \tag{2.2}$$

hold for $x \in (x_0 - \varepsilon; x_0 + \varepsilon) \cap \overline{I}$*.*

Proof. Consider the C^{k-1} mapping

$$\begin{aligned} \Psi \;:\; \overline{I} \times \mathbb{R} \times \mathbb{R} &\longrightarrow \overline{I} \times \mathbb{R} \times \mathbb{R} \\ \begin{pmatrix} x \\ z \\ p \end{pmatrix} &\longmapsto \begin{pmatrix} x \\ z \\ F_p(x, z, p) \end{pmatrix} \end{aligned}$$

By strict convexity of F in p we can see that Ψ is injective and its Jacobian is positive definite. The implicit function theorem provides an inverse mapping of same regularity thus proving that Ψ is a C^{k-1} diffeomorphism with its image. We associate with any solution u of (2.1) a C^{k-1} mapping $\widetilde{u}$ defined on a subset of $J \subset \overline{I}$:

$$\begin{aligned} \widetilde{u} \;:\; J &\longrightarrow J \times \mathbb{R} \times \mathbb{R} \\ x &\longmapsto \begin{pmatrix} x \\ u(x) \\ u'(x) \end{pmatrix} \end{aligned}$$

for which the following first order differential equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\,[\widetilde{u}]^{(x)} &= 1 \\ \frac{\mathrm{d}}{\mathrm{d}x}\,[\widetilde{u}]^{(z)} &= [\widetilde{u}]^{(p)} \\ \frac{\mathrm{d}}{\mathrm{d}x}\,F_p(\widetilde{u}) &= F_z(\widetilde{u}) \end{aligned} \tag{2.3}$$

holds; where $[\widetilde{u}]^{(\cdot)}$ stands for the $\cdot$ coordinate of $\widetilde{u}$. Applying Ψ to $\widetilde{u}$ we obtain $\widetilde{v} := \Psi \circ \widetilde{u} = \begin{pmatrix} x \\ z(x) \\ q(x) \end{pmatrix}$, a solution to the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} x \\ z \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ \left[\Psi^{-1}(x,z,q)\right]^{(p)} \\ F_z\left(\Psi^{-1}(x,z,q)\right) \end{pmatrix} \tag{2.4}$$

Vice versa let $J \subset \overline{I}$ be an open interval and $\widetilde{v} : J \to \Psi(J \times \mathbb{R} \times \mathbb{R})$ be a C^k solution of (2.4), by applying Ψ^{-1} to it, gives a C^{k-1} mapping $\widetilde{u}(x) = \Psi^{-1} \circ \widetilde{v}(x)$ for which (2.3) holds. Furthermore thanks to the second equality in (2.3) $\frac{\mathrm{d}}{\mathrm{d}x}\,[\Psi^{-1} \circ \widetilde{v}]^{(z)} = [\Psi^{-1} \circ \widetilde{v}]^{(p)}$ and so $u(x) = [\Psi^{-1} \circ \widetilde{v}(x)]^{(z)}$ is actually C^k and satisfies (2.1). The initial conditions are transformed accordingly.

We have proved that there is a one-to-one correspondence between C^k solutions to (2.1) and C^k solutions (2.4). This follows from the fact that any C^{k-1} solution to (2.4) are actually C^k.

The dependence of the C^k solutions to (2.4) on the initial data is C^{k-2} as given by standard results for ODE's (the reader may refer to [9]). The class $k-2$ and not $k-1$ is due to the fact that the necessary Lipschitz condition may fail for F_p, F_z and Ψ. The initial conditions for (2.4) themselves are the image by Ψ of the initial values for (2.1).

Applying the Cauchy's existence and uniqueness theorem and taking into account the dependence on the initial conditions we have the existence and uniqueness of a C^k solution to (2.1) with C^{k-2} dependence on initial conditions. Since A is bounded, it is contained in a compact set inside the domain of F. If $k \geq 3$ and thus the dependence on initial data conditions is at least C^1, making the necessary estimates to prove (2.2) is straightforward.

$\square$

2.2 Classical results of variational calculus reproduced in the AC settings

Now we focus on proving a variant of Hilbert's theorem that will guarantee us the minimizing property of a local extremal among AC functions with fixed boundary conditions. The difference from the classical result is the extension to AC which is done by approximation. However it is important to notice that it is necessary to extend the definition of

$\mathscr{F}$ to the class of AC functions. It can be proved that the functional is well defined by setting it to be the integral of the composite function $F(x, u(x), u'(x))$ for any AC function u. In fact the measurability of the integrand is a standard result on the composition of measurable and continuous functions that can be found in the appendix in theorem 6.1. The integrability of the positive and negative parts, on the other hand, depends specifically on the regularity and convexity properties of the Lagrangian and we now give a short proof of this result.

Lemma 2.4. *Let $\mathscr{F}$ be a functional with a C^1 Lagrangian $F(x, z, p)$ on $\overline{I} \times \mathbb{R} \times \mathbb{R}$. Let $F(x, z, p)$ be convex in p i.e. $\forall (x, z) \in \overline{I} \times \mathbb{R} \quad p \mapsto F(x, z, p)$ is convex. Then*

$$\mathscr{F}(u) = \int_0^1 F(x, u(x), u'(x))\,\mathrm{d}x$$

is well defined for $u \in AC(I)$. In particular the integrand is either summable or has a summable negative part and a positive part with an infinite integral.

Proof. Consider the function $f : \overline{I} \to \mathbb{R}$ defined to be $f(x) \mapsto F_p(x, u(x), 0)$ for a given function $u \in AC(I)$. f is continuous since F_p and u are continuous and the domain of f is compact so f attains maximum and minimum M, m on the domain. $F(x, u(x), 0)$ also attains its minimum b on $\overline{I}$. We can now make the following estimate for the integrand:

$$F(x, u(x), u'(x)) > \begin{cases} b + Mu'(x) & \text{if } u'(x) < 0 \\ b + mu'(x) & \text{if } u'(x) > 0 \end{cases}$$

so we have that the integrand is bounded from below by affine transforms of the L^1 function u' that are also L^1 functions. $\square$

Theorem 2.5 (Hilbert's Theorem)**.** *Let $\mathscr{F}$ be a functional with a C^2 Lagrangian $F(x, z, p)$ on $\overline{I} \times \mathbb{R} \times \mathbb{R}$. Let $F(x, z, p)$ be convex in p i.e. $\forall (x, z) \in \overline{I} \times \mathbb{R} \quad p \mapsto F(x, z, p)$ is convex. Let $u_0 \in C^1(\overline{I})$ be an extremal. Suppose that u_0 is embedded in a C^1 field of extremals defined in an open, simply connected neighborhood D of the (x, z) graph of u_0. By this we mean that there exists $\Phi(x, z) \in C^1(D; \mathbb{R})$ such that if a certain $v \in C^1$ satisfies*

$$v'(x) = \Phi(x, v(x)) \quad \textit{locally} \tag{2.5}$$

then v locally satisfies the Euler equation (2.1). Furthermore by being embedded in this field we mean that u_0 is actually a solution of (2.5).

Then u_0 is a minimizer of $\mathscr{F}$ on $\mathcal{C} = \{u \in AC(\overline{I}), \mathrm{Graph}(u) \subset D, u = u_0$ on $\partial I\}$ where $\mathscr{F}$ is defined by (2.4). If convexity is strict then the minimizer is also strict.

Proof. Let $E(x, z, p, q)$ be the Weierstrass function for F i.e.

$$F(x, z, q) = F(x, z, p) + F_p(x, z, p)(q - p) + E(x, z, p, q).$$

$E(x, z, p, q) \geq 0$ for all x, z, p, q by convexity of F. The inequality is strict if convexity is strict and if $p \neq q$. Now we write for any $u \in C^1(\overline{I})$ with graph in D:

$$\begin{aligned}
\mathscr{F}(u) &= \int_a^b F(x,u(x),u'(x))\,\mathrm{d}x = \\
&= \int_a^b E(x,u,\Phi(x,u),u')\,\mathrm{d}x + \int_a^b \big(F(x,u,\Phi(x,u)) + F_p(x,u,\Phi(x,u))(u'-\Phi(x,u))\big)\ \mathrm{d}x \geq \\
&\geq \int_a^b \big(F(x,u,\Phi(x,u)) + F_p(x,u,\Phi(x,u))(u'-\Phi(x,u))\big)\ \mathrm{d}x = \\
&= \int_a^b \big(F(x,u,\Phi(x,u)) - \Phi(x,u)F_p(x,u,\Phi(x,u)) + F_p(x,u,\Phi(x,u))u'\big)\ \mathrm{d}x \qquad (2.6)
\end{aligned}$$

We observe that the last integral is the integral of the C^1 differential form

$$\omega = \big(F(x,z,\Phi(x,z)) - \Phi(x,z)F_p(x,z,\Phi(x,z))\big)\,dx + F_p(x,z,\Phi(x,z))\,\mathrm{d}z$$

along the graph of u. Consider a solution v to (2.5) with graph contained in D. Since $\Phi \in C^1(D)$ we have that $v \in C^2$ so for any point $(x,v(x),v'(x))$ we can apply the chain rule to the left part of Euler's equation (2.1):

$$F_{px} + v'F_{pz} + v''F_{pp} = F_z.$$

Differentiating both sides of (2.5) we get

$$v''(x) = \Phi_x(x,v(x)) + \Phi(x,v(x))\Phi_z(x,v(x))$$

and putting these relations together, since by hypothesis v satisfies Euler's equation, we get for any point $(x,v(x))$:

$$F_{px} + \Phi F_{pz} + (\Phi_x + \Phi\Phi_z)F_{pp} = F_z \qquad (2.7)$$

where F and its derivatives are evaluated in $\big(x,v(x),\Phi(x,v(x))\big)$. This reasoning remains valid for any solution to (2.5). Φ is defined for all points of D so for any point there is a solution that passes through it. Consequently (2.7) is valid everywhere on D.

Since ω is C^1 we can calculate $\mathrm{d}\omega$ and by differentiating its coefficients and together with (2.7) we get $\mathrm{d}\omega = 0$ i.e. ω is closed. Since D is simply connected ω is exact. We deduce that the right hand term of (2.6) doesn't depend on the particular C^1 function but only on its boundary values so for any $u \in C^1 \cap \mathcal{C}$ we have

$$\int_{u_0} \omega = \int_u \omega.$$

Now we consider $u \in \mathcal{C}$. If $\mathscr{F}(u) = +\infty$ then there is nothing to prove, otherwise we write as in (2.6):

$$\mathscr{F}(u) = \int_a^b E(x,u,\Phi(x,u),u')\,\mathrm{d}x + \int_a^b F(x,u,\Phi(x,u)) + F_p(x,u,\Phi(x,u))(u'-\Phi(x,u))\,\mathrm{d}x$$

where both integrals are well defined for a reasoning similar to (2.4), since all the terms are convex in u' and are continuous. Finally, to deal with the second integral term we choose $\widetilde{u}_n \in \mathcal{C} \cap C^1$ with $\widetilde{u}_n \xrightarrow{\|\cdot\|_{AC}} u$ and with all their (x, z) graphs contained in a compact subset $K \subset D$. Since $\widetilde{u}_n \xrightarrow{\|\cdot\|_\infty} u$, the F, F_p and Φ are uniformly continuous on K and $\widetilde{u}'_n \xrightarrow{L_1} u'$ we have that

$$\int_{u_0} \omega = \int_{\widetilde{u}_n} \omega \to$$
$$\to \int_a^b \big(F(x, u, \Phi(x, u)) - \Phi(x, u) F_p(x, u, \Phi(x, u)) + F_p(x, u, \Phi(x, u)) u'\big)\, \mathrm{d}x.$$

Thus for all $u \in \mathcal{C}$

$$\mathscr{F}(u) = \int_a^b E(x, u, \Phi(x, u), u')\, \mathrm{d}x + \mathscr{F}(u_0) \geq \mathscr{F}(u_0)$$

since E is non-negative. The inequality is strict if convexity is strict and $u \neq u_0$. □

2.3 Proof of Tonelli's partial regularity theorem in the regular case

We are now ready to prove Tonelli's partial regularity theorem.

Proof of theorem (2.1).

Part 1 We will show that for all points y_0 where

$$L := \limsup_{y \to y_0} \left| \frac{w(y) - w(y_0)}{y - y_0} \right| < \infty \tag{2.8}$$

we can choose an interval containing y_0 small enough to find a solution v of Euler's equation coinciding with w at the end points by using lemma 2.3. If the interval was chosen small enough, the function $\widetilde{w}$ obtained by substituting w by v on the interval belongs to the AC neighborhood on which w is a minimizer (we consider the $W^{1,1}$ norm on AC). Again by using lemma 2.3 we will also exhibit a C^1 field of extremals, in which v will be embedded, that will be defined on a simply connected open region containing the graph of w over the interval. Hilbert's theorem states that v is a strict local minimizer of the functional restricted to the small interval so the value of the functional on $\widetilde{w}$ would be less than that of w. This would contradict the minimality property of w so w and $\widetilde{w}$ must coincide. This provides us with the regularity of w on the small interval.

We start by choosing $\sigma = \frac{|L|}{4}$ and noticing that for any $\varepsilon > 0$ we can choose points $x_0 < y_0 < x_1$ arbitrarily close to y_0, so that $|x_1 - x_0| < \varepsilon$, for which

$$\left| \frac{w(x_1) - w(x_0)}{x_1 - x_0} \right| < L + \sigma. \tag{2.9}$$

We will now choose $N > 0$ so as to be able to find a suitable solution v to Euler's equation of $[x_0, x_1]$ among those provided by the lemma 2.3. The lemma will be applied starting from the point $(x_0, w(x_0))$ and v must satisfy $v = w$ on the boundary points. We will need to choose $M > 0$ so as to be able to find the above mentioned C^1 field of extremals. To do this set $\delta = \frac{\sigma}{2}$, $N = L + 2\sigma + \delta$. Then choose $M = \frac{1}{1-\widetilde{\delta}}\left(\max_{\overline{I}}(w) - \min_{\overline{I}}(w) + N + \delta\right)$ having fixed $\widetilde{\delta} = \frac{1}{2}$ from now on. To apply lemma 2.3 we choose $A = \overline{I} \times (\min(u_0) - 1, \max(u_0) + 1)$. The lemma provides us with an $\varepsilon > 0$ and a family of solutions u so that the estimates in (2.2) hold.

We will now impose further conditions on ε so that a solution v chosen by suitably applying lemma 2.3 will be in the AC neighborhood of w on which w is a minimizer.

Being w a local AC minimizer there exists a $d > 0$ such that w is a minimizer among all competitors $\widetilde{w} \in \mathcal{C}$ such that $\|w - \widetilde{w}\|_{AC} < d$. We can impose $\varepsilon > 0$ small enough for the inequalities

$$\begin{aligned} \sup_{\substack{r,s\in[0,1]\\|r-s|<\varepsilon}} |w(r) - w(s)| &< \frac{d}{4} \\ \sup_{\substack{r,s\in[0,1]\\|r-s|<\varepsilon}} \int_r^s |w'| &< \frac{d}{4} \\ (N+\delta)\varepsilon &< \frac{d}{4}. \end{aligned} \tag{2.10}$$

to hold. The first inequality is possible since w is uniformly continuous while the second one is due to the absolute continuity property of the integral of L_1 functions. Once we choose the solution v applying lemma 2.3 in part two we will show in part three that v lies in the correct neighborhood of w

Part 2 We now fix $x_0 < y_0 < x_1$, $|x_1 - x_0| < \varepsilon$ with ε provided by lemma 2.3 and satisfying (2.9) and (2.10). In this part we show that it is possible to choose from the solutions given by the lemma a solution v with staring point $(x_0, w(x_0))$ that coincides with w on the boundary points of $[x_0, x_1]$.

Consider the family of solutions with starting point $(x_0, w(x_0))$ given by the lemma and consequently the mapping of $(-L, L)$ given by $(b) \mapsto u(x_1; 0, b)$. We can find a $|\widetilde{b}| < N$ for which $u(x_1; 0, \widetilde{b}) = w(x_1)$. This is true since by (2.9) we have that

$$w(x_0) - (L+\sigma)(x_1 - x_0) < w(x_1) < w(x_0) + (L+\sigma)(x_1 - x_0)$$

while integrating the first inequality in (2.2) we get that:

$$\begin{aligned} u(x_1; 0, N) &> w(x_0) + (x_1 - x_0)(N - \delta) = w(x_0) + (x_1 - x_0)(L + 2\sigma) \\ u(x_1; 0, -N) &< w(x_0) - (x_1 - x_0)(N - \delta) = w(x_0) - (x_1 - x_0)(L + 2\sigma). \end{aligned}$$

By continuity of the dependence of u on b we have the required result. We thus define $v(x) = u(x; 0, \widetilde{b})$ for $x \in [x_0, x_1]$.

Part 3 Here we show that the choice of v done in the previous part together with inequalities (2.10) give us v in a sufficiently close vicinity of w to be able to subsequently apply reasonings based on the minimizing property of w.

Recall that d was defined so that w is a minimizer among all competitors $\widetilde{w} \in \mathcal{C}$ such that $\|w - \widetilde{w}\|_{AC} < d$. Consider the solution $v(x) = (x;\, 0, \widetilde{b})$ found with the above mentioned procedure. It satisfies $\|w - v\|_{AC} < d$ on $[x_0, x_1]$. It is possible to verify this using the first inequality in (2.2) to get

$$\left\|w' - v'\right\|_{L_1} \leq \int_{x_0}^{x_1} |w'| + \int_{x_0}^{x_1} |v'| < \frac{d}{4} + (L+\delta)\varepsilon < \frac{d}{2}.$$

On the other hand using the same inequality form (2.2) and the Lagrange theorem we get

$$\begin{aligned}\|w - v\|_\infty &\leq \left\|w - w(x_0)\right\|_\infty + \left\|v - w(x_0)\right\|_\infty < \\ &< \frac{d}{4} + \left\|v(x) - v(x_0)\right\|_\infty < \frac{d}{4} + (L+\delta)\varepsilon < \frac{d}{2}\end{aligned}$$

This gives us the desired result $\|w - v\|_{AC} < \|w' - v'\|_{L_1} + \|w - v\|_\infty < d$.

Part 4 In the previous parts we established an interval $[x_0, x_1]$ and a solution v to Euler's equation coinciding with w on the boundary of this interval. Given our initial choice of M, in this part we will exhibit a C^1 extremal field defined on a simply connected open region containing the graphs of v and w over the interval. Finally we will apply Hilbert's theorem to w and v on $[x_0, x_1]$ to deduce by the minimizing property of w that v and w must coincide. Lemma 2.3 provides w with the needed regularity on the interval.

Having fixed $\widetilde{b}$ in the previous parts, let's vary a in $u(x;\, a, \widetilde{b})$ to cover a region containing the graphs of v and w over $[x_0, x_1]$ with a C^1 field of extremals. We define the field $\Phi(x, z)$ through the implicit equation

$$\Phi\left(x, u(x;\, a, \widetilde{b})\right) := u'(x;\, a, \widetilde{b}). \tag{2.11}$$

As a matter of fact, the second inequality of (2.2) gives us that $\partial_a u(x;\, a, \widetilde{b}) > 1 - \widetilde{\delta} > \frac{1}{2}$. This proves that $(a) \mapsto u(x;\, a, \widetilde{b})$ is injective for any fixed $x \in [x_0, x_1]$ so the field is well defined by (2.11).

Integrating both inequalities in (2.2) and bearing in mind the choice of M we have that for any $x \in [x_0, x_1]$

$$\begin{aligned}u(x;\, M, \widetilde{b}) - w(x_0) =& u(x;\, M, \widetilde{b}) - u(x_0;\, 0, \widetilde{b}) \\ & > (1 - \widetilde{\delta})M + u(x;\, 0, \widetilde{b}) - u(x_0;\, 0, \widetilde{b}) > \max_{\overline{I}}(w) - \min_{\overline{I}}(w) \\ u(x;\, -M, \widetilde{b}) - w(x_0) =& u(x;\, -M, \widetilde{b}) - u(x_0;\, 0, \widetilde{b}) \\ & < -(1 - \widetilde{\delta})M + u(x;\, 0, \widetilde{b}) - u(x_0;\, 0, \widetilde{b}) < -\left(\max_{\overline{I}}(w) - \min_{\overline{I}}(w)\right)\end{aligned}$$

Arguing by continuity, we have that for any $x \in [x_0, x_1]$ and for any $z \in [\min(w), \max(w)]$ (x, z) is in the image of the map $(a) \mapsto u(x;\, a, \widetilde{b})$ of the interval $(-M, M)$. Thus Φ, defined by 2.11, is defined on an open simply connected neighborhood $D \subset$

$[x_0, x_1] \times \mathbb{R}$ containing the graph of w; by definition v is embedded in this field. The second inequality of (2.2) provides us with $\partial_a u(x;\, a, \widetilde{b}) \neq 0$, and lemma 2.3 provides us with $u(x;\, a, \widetilde{b})$ that has a C^1 dependence on a. Applying the implicit function theorem to the relation (2.11) gives us that $\Phi \in C^1(D)$ as needed by Hilbert's Theorem.

Finally define

$$\widetilde{w}(x) = \begin{cases} v(x) & \text{if } x \in [x_0, x_1] \\ w(x) & \text{otherwise} \end{cases}. \tag{2.12}$$

From the considerations made in part three we have that $\|w - \widetilde{w}\|_{AC} < d$. We will thus compare w and $\widetilde{w}$ using the minimizing property of w.

Hilbert's theorem 2.5 now gives that the value of the functional restricted to $[x_0, x_1]$ applied to v is strictly smaller than to the value assumed on any different AC function with graph in D and same boundary values; this includes w. Thus, having defined $\widetilde{w}$ in (2.12) we have that

$$\mathscr{F}(\widetilde{w}) < \mathscr{F}(w)$$

but by construction w is a local minimizer on some AC neighborhood and $\widetilde{w}$ lies in that neighborhood. This is absurd so w and $\widetilde{w}$ must coincide.

We have just proved that the minimizer w has to coincide with a solution of Euler's equations, that is C^k, in an open neighborhood of all points where (2.8) holds. Since w is differentiable almost everywhere then we have that w is C^k on an open set Ω_0 of full measure. Furthermore it can be proved by reviewing the steps taken so far that w is actually locally a strict minimizer of $\mathscr{F}$.

Part 5 It remains to prove that on $E = \overline{I} \setminus \Omega_0$ w has an infinite derivative that is continuous in those points. Consider $y_0 \in E$ for which (2.8) doesn't hold. This means that there are $y_j \to y_0$ for which

$$\lim_{j\to\infty} \left| \frac{w(y_0) - w(y_j)}{y_0 - y_j} \right| = \infty.$$

Suppose for now that $y_j < y_{j+1} < y_0$ and the terms of the limit are positive (the other cases are very similar). A priori we do not know what happens on the right (other) side of y_0 but in the course of the proof we will show that the behavior is the same.

We now choose an arbitrarily large $N > 0$, some $M > 0$ and a small $\delta > 0$. As before we consider the field of extremals $u'(y;\, \alpha, N)$ with $\alpha < M$ given by applying the lemma (2.3) at $(y_0, w(y_0))$. We do this as before by considering the extremals obtained by fixing the slope and changing the value $w(y_0) + \alpha$ of the solution. The implicit function theorem, gives as a small open neighborhood $D \subset (y_0 - \varepsilon, y_0 + \varepsilon) \times \mathbb{R}$ that thanks to the estimates of the lemma (2.3) can be chosen to be simply connected. Furthermore, always due to the estimates of the lemma, since $\partial_a u \neq 0$ and $w(y)$ is continuous the implicit function theorem gives us a continuous $\alpha(y)$ for which

$$w(y) = u(y;\, \alpha(y), N) \quad \forall y \in J$$

where J is a small open interval containing y_0. Graphically $\alpha(y)$ is the value in y_0 of the extremal that w intersects in y. We notice that $\alpha(y_0) = 0$ and $\alpha(y)$ is definitely monotonous and non-decreasing. The monotonicity follows from the fact that in a sufficiently small neighborhood of y_0, the graph of $w(x)$ falls into the region where the C^1 field of extremals is defined. If there were $r, s \in J$ in this neighborhood for which $\alpha(r) = \alpha(s)$ it would mean that w intersects the same extremal $u(x, \alpha(r), N)$ twice, once in r and once in s. But by minimality w cannot do this; otherwise it would contradict Hilbert's theorem (2.5) since the integral of the Lagrangian over the extremal is strictly lower that that of any AC function in the region. We have that $\alpha(x)$ cannot be decreasing since otherwise we would have bounded difference quotients for $y_j \to y_0$ as seen from this inequality:

$$\frac{w(y_0) - w(y_j)}{y_0 - y_j} = \frac{u(y_0, 0, N) - u(y_j, \alpha(y_j), N)}{y_0 - y_j} < \frac{u(y_0, 0, N) - u(y_j, 0, N)}{y_0 - y_j} < N + \delta.$$

We have used that u is increasing in a as given by lemma 2.3 and for the last step the Lagrange theorem together with the estimate from the first inequality in (2.2).

Now choose two arbitrary sequences $z_i > x_i$ tending monotonously to y_0. Starting from a certain index j_0, $w(x)$ $x_{j_0} \leq x \leq z_{j_0}$ will have the graph in the region where the field of extremals is defined and α is defined and monotonous. We will thus have

$$\begin{aligned}\frac{w(z_j) - w(x_j)}{z_j - x_j} &= \frac{u(z_j, \alpha(z_j), N) - u(x_j, \alpha(x_j), N)}{z_j - x_j} > \\ &> \frac{u(z_j, \alpha(x_j), N) - u(x_j, \alpha(x_j), N)}{z_j - x_j} > N - \delta.\end{aligned}$$

again by monotonicity of α, u in a and by the Lagrange theorem together with the first inequality in (2.2)

Since N and δ were chosen arbitrarily we conclude that for any choice of sequences (x_n) and (z_n) as above we have definitively $\frac{u(z_j)-u(x_j)}{z_j-x_j} > N - \delta$ for arbitrary N and δ. As a consequence:

$$\lim_{x_j < z_j \to y_0} \frac{u(z_j) - u(x_j)}{z_j - x_j} = +\infty.$$

Hence the existence and continuity of the derivative.

□

3 The singular set

3.1 Mania's example for a non-empty singular set

In [8] Mania gives an example of a functional with a very regular Lagrangian (it is actually a polynomial) that exhibits a minimizer for which the singular set is non-empty. It consists of one bounadry point where the minimizer has an infinite derivative. The Lagrangian doesn't actually satisfy the hypothesis of Tonelli's theorem since the convexity assumption fails exactly on the minimizer. The example is nevertheless important as a slightly more complex construction based on the same ideas can be used to exhibit Tonelli-like functionals with arbitrary closed Lebesgue null-measure singular sets. Mania's example also exhibits the Lavrentiev phenomenon which is central to the study of singular functionals.

Example 3.1 (Mania's example)**.** *Consider the functional* $\mathscr{F}(u) = \int_0^1 F(x, u, u')\,\mathrm{d}x$ *with* $F(x, z, p) = (z^3 - x)^2 p^6$ *on functions of class* $\mathcal{C} = \{u \in AC([0,1]) \mid u(0) = 0,\ u(1) = 1\}$*. The only minimizer is* $v(x) = x^{\frac{1}{3}}$ *that has a singularity at* $x = 0$*. The Lavrentiev phenomenon occurs for* $\mathscr{F}$*, i.e.*

$$\inf_{u \in \mathrm{Lip}([0,1]) \cap \mathcal{C}} \mathscr{F}(u) > \mathscr{F}(v).$$

Furthermore this particular example has the property that if $v_n \rightrightarrows v$ *uniformly in* $[0,1]$ *and* $v_n \in \mathrm{Lip}([0,1]) \cap \mathcal{C}$*, then* $\mathscr{F}(v_n) \to +\infty$*.*

Proof. It is easy to see that $\mathscr{F}(u) \geq 0$ and $\mathscr{F}(x^{\frac{1}{3}}) = 0$. Trivially no other minimizer can exist. As a matter of fact if $u(x) \neq v(x)$ for some $x \in (0,1)$, $u' \neq 0$ for some set of non-zero measure contained in the interval containing x on which $u \neq v$. The integrand is non-negative and it is thus strictly positive on that set.

The idea is to compare v_n with v, $\frac{1}{2}v$ and $\frac{1}{4}v$. For an approximating sequence of functions this comparison will be definitely done in a neighborhood of 0 since $v_n \xrightarrow{\|\cdot\|_\infty} v$. Consider some function $u \in \mathrm{Lip}([0,1]) \cap \mathcal{C}$ and let $a = \max\{x \mid u(x) = \frac{1}{4}v(x)\}$ and $b = \min\{x > a \mid u(x) = \frac{1}{2}v(x)\}$ so:

$$\frac{1}{4}x^{\frac{1}{3}} < u(x) < \frac{1}{2}x^{\frac{1}{3}} \quad \text{for } x \in [a, b]. \tag{3.1}$$

We will now give a lower bound on $\int_a^b F(x, u, u')\,\mathrm{d}x$. First of all (3.1) gives us $(u^3 - x)^2 > \left(\frac{7}{8}\right)^2 x^2 > \left(\frac{7}{8}\right)^2 a^2$ in the area we are dealing with. By Hölder's inequality we have

$$\int_a^b u'^6 > \frac{(\int_a^b |u'|)^6}{(b-a)^5} > \frac{|u(b) - u(a)|^6}{(b-a)^5} > a^{-3}\left(\frac{1}{4}\right)^2 \frac{\left(2(\frac{b}{a})^{1/3} - 1\right)^6}{(\frac{b}{a} - 1)^5}.$$

Combining these two estimates we have:

$$\int_a^b (u^3 - x)^2 u'^6\,\mathrm{d}x > a^{-1}\left(\frac{7}{8}\right)^2 \left(\frac{1}{4}\right)^2 \frac{\left(2(\frac{b}{a})^{1/3} - 1\right)^6}{(\frac{b}{a} - 1)^5}.$$

If $b \leq 2a$ we have $\frac{\left(2(\frac{b}{a})^{1/3} - 1\right)^6}{(\frac{b}{a} - 1)^5} > 1$ while if $b > 2a$ we can change the interval that we consider to $[a, 2a]$ and Hölder's inequality will give us

$$\int_a^{2a} u'^6 > (\int_a^{2a} |u'|)^6 > \frac{|u(2a) - u(a)|^6}{(2a-a)^5} > a^{-3}\left(\frac{1}{4}\right)^2 \left(2^{1/3} - 1\right)^6.$$

In this case we have

$$\int_a^b (u^3 - x)^2 u'^6 \, \mathrm{d}x > a^{-1} \left(\frac{7}{8}\right)^2 \left(2^{1/3} - 1\right)^6.$$

The proof is now complete since for any u we have that $a^{-1} > 1$ whereas if $v_n \xrightarrow{\|\cdot\|_\infty} v$ the corresponding $a_n^{-1} \to +\infty$ $\square$

One might think that the origin of the Lavrentiev phenomenon in this example is given by the lack of super-linearity or strict convexity. In fact these two conditions are typical for existence, regularity and semi-continuity results (6.2) and these conditions fail exactly on the minimizer. However one can see that this is not central to the example. It suffices to consider that $\int_0^1 |v'|^\alpha \, \mathrm{d}x$ converges for the minimizer $v(x) = x^{\frac{1}{3}}$ and $0 < \alpha < 3/2$. Given $\alpha > 1$, let $\varepsilon > 0$ such that

$$\varepsilon \int_0^1 \left|v'\right|^\alpha < \inf_{u \in \mathrm{Lip}([0,1]) \cap \mathcal{C}} \mathscr{F}(u).$$

Modifying the functional $\mathscr{F}$ by adding this term we set

$$\widetilde{\mathscr{F}}(u) = \int_a^b \min(4, (u^3 - x)^2) u'^6 + \varepsilon |u'|^\alpha \, \mathrm{d}x.$$

to have a uniformly strictly convex functional with a Lagrangian with controlled growth conditions:

$$\varepsilon |p|^\alpha < \widetilde{F}(x, z, p) < c_1 |p|^6 + c_2.$$

By making this modification we have lost regularity but we will see in the next section that it is possible to have smooth convex Lagrangians with super-quadratic growth order in p that exhibit similar behavior to Mania's example.

3.2 Optimality of the description of the singular set

We will now prove that the description of the singular set given by Tonelli's theorem is optimal. In particular for any closed $E \subset I = [0,1]$ of null Lebesgue measure we construct a smooth strictly convex Lagrangian a minimizer of which on $\mathcal{C} = \{u \in AC \mid u(0) = 0,\, u(1) = 1\}$ must contain E as its singular set. Finally we will prove that a minimizer can have no singularity points outside E.

The ideas of the proof is due to Davie [4] and is similar to the ones used in Mania's example. We will start by considering a Lagrangian on made of two terms:

$$F(x, u, u') = \left(\varphi(u) - \varphi(v(x))\right)^2 \psi(u') + u'^2. \tag{3.2}$$

The second term is simply quadratic and thus is smooth and provides strict convexity everywhere. The first term is a penalty function the construction of which is the central part of the proof. The form of the penalty function is a generalization of Mania's example for which the choices were $v(x) = x^{\frac{1}{3}}$, $\varphi(v) = v^3$ and $\psi(p) = p^6$.

Reasoning by analogy we first choose $v(x) \in \mathcal{C}$ to be a target minimizer that has the desired properties, in particular it should be $C^\infty(I \setminus E)$ and should have a continuous derivative that is infinite for the points of E. We also need $v' \in L^2$ to make the p^2 term converge and we choose v strictly increasing. The function φ should be chosen to be $C^\infty(I)$ and such that $\varphi \circ v \in C^\infty(I)$ since the penalty functional must be regular. We will also ask $\varphi \circ v$ to be strictly increasing to make some necessary estimates possible. Finally we will choose ψ to be a non negative C^∞ function with $\psi(0) = 0$ and with sufficient growth to make an estimate similar to the one in Mania's example possible i.e. we will ask that if $u \in \mathcal{C}$ and u' is finite for a point in E then the integral of the penalty function over some interval close by can be controlled from below by a constant. Rescaling ψ this constant can then be made arbitrarily large and in particular larger than $\int_0^1 v'^2$. We notice that ψ must also be chosen convex for the functional to be convex. Finally we will prove that outside E the minimizer doesn't have singularities since have not proven that v will be the actual minimizer. Since the proof depends on comparing v to a candidate minimizer, it is important to notice that our example gives a Lagrangian satisfying the hypothesis for the existence of an AC minimizer (6.2).

Theorem 3.2 (Davie's example for minimizers with arbitrary singular sets satisfying Tonelli's partial regularity theorem)**.** *Let I be the unit interval and let $E \subset \overline{I}$ be a closed set of zero Lebesgue measure. We can find functions φ and ψ and a function $v \in \mathcal{C} = \{u \in AC \mid v(0) = 0, v(1) = 1\}$ such that the Lagrangian (3.2)*

$$F(x, z, p) = \big(\varphi(z) - \varphi(v(x))\big)^2 \psi(p) + p^2$$

satisfies the hypothesis of Tonelli's regularity (2.1) and Tonelli's existence theorems. In particular $F(x, z, p)$ is of class $\mathbb{C}^\infty(\overline{I} \times \mathbb{R} \times \mathbb{R})$ and $F_{pp} > 0$ and is super-linear in p. Furthermore for the associated functional

$$\mathscr{F}(u) = \int_0^1 F(x, u, u')\,\mathrm{d}x$$

we have:

1. *$\mathscr{F}$ is defined on $\mathcal{C}$ with values in $\mathbb{R} \cup \{+\infty\}$.*
2. *$\mathscr{F}$ attains its infimum on $\mathcal{C}$.*
3. *If u is any minimizer of $\mathscr{F}$ then the singular set of u is exactly E*
4. *The Lavrentiev phenomenon occurs i.e.*

$$\inf_{\mathcal{C}} \mathscr{F} < \inf_{\mathrm{Lip} \cap \mathcal{C}} \mathscr{F}$$

Proof. **Part 1** In this part we construct a suitable function v that is C^∞ with continuous extended derivative, for which $v' = +\infty$ exactly on E

Approximate E externally with a sequence of open sets

$$A_n = \{x \in \overline{I} \mid d(x, E) < \varepsilon_n\} \tag{3.3}$$

by choosing $\varepsilon_n \to 0$ strictly decreasing, such that $\mathscr{L}(A_n) < 2^{-n}$ and $\overline{A}_{n+1} \subset A_n$. Since E is closed, $\bigcap_{n\in\mathbb{N}} A_n = E$.

Consider bump functions $g_n \in C^\infty(\overline{I})$ for which $g_n = 1$ on A_{n+1}, $g_n = 0$ on A_n^c and $0 \le g_n \le 1$. Consider

$$g = 1 + \sum_{n=0}^{\infty} g_n \tag{3.4}$$

and

$$v(x) = \gamma^{-1} \int_0^x g(t)\,\mathrm{d}t.$$

We have that $g \in L_2$ so $v \in AC$ and setting $\gamma = \|g\|_{L_1}$ we have $v(0) = 0$ and $v(1) = 1$. We also have

$$\begin{gathered} v'(x) > \gamma^{-1} \quad \forall x \in \overline{I} \\ \text{and} \\ \gamma^{-1} n \le v'(x) \le \gamma^{-1}(n+1) \quad \text{for } x \in A_n \setminus A_{n+1}. \end{gathered} \tag{3.5}$$

v is then strictly increasing and $v' = +\infty$ on E. For all $x \notin E$ the non zero terms in (3.4) are finite so v is C^∞ on $\overline{I} \setminus E$. Also, v' is continuous in $\mathbb{R} \cup \{+\infty\}$

Part 2 In this part we construct a strictly increasing regular function φ such that $\varphi(v)$ is also C^∞.

Let's notice that $S = v(E) \subset [0,1]$ is closed and nowhere dense. Being closed is due to the fact that v is strictly increasing. $\mathring{S} = \emptyset$ since otherwise E would contain an interval and wouldn't be of zero measure. Consider an approximating sequence for S^c of closed sets $B_n = \{x \in [0,1] \mid d(x,S) \ge \delta_n\}$ by choosing $\delta_n \to 0$ so that $B_n \subset \mathring{B}_{n+1}$. Consider bump functions $0 \le \chi_n \le 1$ that are $\chi_n = 1$ on B_n and $\chi_n = 0$ on $\overline{B^c}_{n+1}$. Let $f_n(y) = \int_{-2}^{y} \chi_n(t)\,\mathrm{d}t$ Now choose β_n so that for all $1 \le k \le n$

$$\begin{gathered} \beta_n \left| D^{(k)} f_n(y) \right| < 2^{-n} \quad \forall y \in \mathbb{R} \\ \beta_n \left| D^{(k)} \left[f_n(v(x)) \right] \right| < 2^{-n} \quad \forall x \in \overline{I}. \end{gathered}$$

This is possible since each χ_n was chosen to have a compact support and f_n are constant on a small neighborhood of S. Let

$$\varphi(y) = \sum_{n=0}^{+\infty} \beta_n f_n(y).$$

φ and every one of its derivatives, converge uniformly to a C^∞ function. Since S is nowhere dense, φ is strictly increasing; so is $\varphi(v)$.

Part 3 In this part we choose a suitable ψ depending only on v and φ that were exhibited in the previous two parts. We need ψ to be regular, convex, non-negative and with $\psi(0) = 0$. We exhibit an appropriate function $\psi(p)$ to make the penalty function sufficiently large for any minimizing competitor that doesn't have a singularity in at least one point of E. In particular we will exhibit ψ such that the integral of the penalty function over a suitably chosen neighborhood of $x_0 \in E$ is larger than $\mathscr{F}(v)$ if $u'(x_0) < +\infty$.

Let us consider a competitor function $u \in \mathcal{C}$ for being a minimizer and suppose that $u' < \infty$ in some point of $x_0 \in E$. It is worth noting that $u' \geq 0$ since if u assumed the same value twice on the end points of some interval it would be possible to strictly reduce the value of the functional by substituting u by a constant on that interval. This implies that $u(x) \in [0,1]$ for $x \in [0,1]$

Notice that since ψ cannot not depend on x_0 we will need to construct ψ in such a way to control the behavior of v in the vicinity of any point of E.

Let us consider the case where $u(x_0) \leq v(x_0)$ and $x_0 < 1$. The other case is very similar.

We proceed as in Mania's example comparing $u(x) - v(x_0)$ with $v(x) - v(x_0)$, $\frac{1}{2}\big(v(x) - v(x_0)\big)$ and $\frac{1}{4}\big(v(x) - v(x_0)\big)$. Bearing in mind that u and v coincide for $x = 1$ and $v'(x_0) = +\infty$ while $u'(x_0)$ is finite, as in Mania's example, we can find $[a,b] \subset (x_0, 1)$ such that

$$\begin{aligned} u(x) < v(x) \quad \forall x \in (a,b) \\ \frac{1}{4}\big(v(x) - v(x_0)\big) <& u(x) - v(x_0) < \frac{1}{2}\big(v(x) - v(x_0)\big) \\ \frac{1}{4}\big(v(a) - v(x_0)\big) =& u(a) - v(x_0) \\ u(b) - v(x_0) = \frac{1}{2}\big(v(b) - v(x_0)\big) \end{aligned} \tag{3.6}$$

and from now on we will try to estimate the integral of the penalty function on $[a,b]$ and all the subsequent estimates are carried out on this interval.

From the first two lines of (3.6) we get an estimate independent of u:

$$v(x) - u(x) > \frac{1}{2}\big(v(x) - v(x_0)\big) > \frac{x - x_0}{2\gamma}.$$

The second inequality comes from the estimates (3.5) of the growth of v. We naturally try to eliminate the the dependence on the absolute position of x_0. Let us introduce the function

$$\eta(d) = \inf_{\substack{x,y \in [0,1] \\ |x-y| \geq d}} \big|\varphi(x) - \varphi(y)\big|^2.$$

Since φ is uniformly continuous on the compact $[0,1]$ the infimum is reached and η is continuous. We have that $\eta(0) = 0$ and the strict monotonicity of φ gives us that η is strictly increasing. Using η we write:

$$\big|\varphi(u(x)) - \varphi(v(x))\big|^2 > \eta\left(\frac{x - x_0}{2\gamma}\right). \tag{3.7}$$

Looking back at the integral of the penalty function we have to estimate the right hand side of the following inequality:

$$\int_a^b \big|\varphi(u(x)) - \varphi(v(x))\big|^2 \psi(u'(x))\,\mathrm{d}x > \int_a^b \eta\left(\frac{x - x_0}{2\gamma}\right)\psi(u'(x))\,\mathrm{d}x$$

If we were to prove that ψ can be chosen so that the following inequality for the integrand

$$\eta\left(\frac{x-x_0}{2\gamma}\right)\psi(u'(x)) > \frac{u'(x)}{(x-x_0)} \tag{3.8}$$

x holds, then we could conclude by using (3.6) and (3.5):

$$\begin{aligned}\int_a^b \frac{u'(x)}{(x-x_0)}\,\mathrm{d}x &> \frac{1}{b-x_0}\int_a^b u'(x)\,\mathrm{d}x = \\ &= \frac{1}{b-x_0}\big(u(b)-u(a)\big) = \frac{1}{b-x_0}\left(\frac{2v(b)-v(a)-v(x_0)}{4}\right) > \\ &> \frac{1}{4}\frac{v(b)-v(x_0)}{b-x_0} > \frac{1}{4\gamma}\end{aligned} \tag{3.9}$$

where the first inequality of the chain is valid since a candidate minimizer has $u' \geq 0$.

However we cannot expect the estimate (3.8) to hold everywhere since the convexity and regularity requirements on ψ constrain it to be sub-linear for small values of u'. We must separate the region where u' is close to 0 from the rest. Consider the set $H = \{t \in (a,b) \mid u'(t) < \frac{v'(t)}{4}\}$ where u' is small. The contribution of the integral of u' over H is small:

$$\int_H u'(x)\,\mathrm{d}x < \frac{v(b)-v(a)}{4}$$

so from (3.6)

$$\int_{(a,b)\setminus H} u'(x)\,\mathrm{d}x > \frac{v(b)-v(x_0)}{4} \tag{3.10}$$

We now need to exhibit a function ψ satisfying

$$\psi(u'(x)) \geq \frac{u'(x)}{(x-x_0)\eta\left({x-x_0}/{2\gamma}\right)}$$

We introduce a function $\rho(x)$ defined in a manner to account for the behavior of v' close to the singular set. Set

$$\rho(\varepsilon_n) = n+1 \tag{3.11}$$

extending it to be affine on the intervals $(\varepsilon_{n+1}, \varepsilon_n]$ and constantly 1 on $(\varepsilon_0; +\infty)$ So as to have a continuous function on $(0,+\infty)$ strictly decreasing for $(0,\varepsilon_0)$. From (3.5) we have

$$v'(x) \geq \gamma^{-1}\rho(|x-x_0|) \tag{3.12}$$

for $x \in \overline{I}$ and any $x_0 \in E$. $\rho^{-1}(y)$ is well defined for $y > 1$, is continuous and decreasing. Finally define

$$\widetilde{\psi}(p) = \begin{cases} \frac{p}{\rho^{-1}(4\gamma p)\eta\left(\frac{\rho^{-1}(4\gamma p)}{2\gamma}\right)} & \text{if } p \geq \frac{1}{4\gamma} \\ 0 & \text{if } p \leq \frac{1}{16\gamma} \\ \text{affine interpolation} & \text{if } \frac{1}{16\gamma} < p < \frac{1}{4\gamma}, \end{cases} \tag{3.13}$$

so that $\widetilde{\psi}$ is continuous and increasing. The denominator term is decreasing and so when $u'(x) \geq \frac{1}{4}v'(x)$ we have, bearing in mind (3.12)

$$\widetilde{\psi}(u'(x)) > \frac{u'(x)}{(x-x_0)\eta\left({}^{x-x_0}\!/_{2\gamma}\right)} \tag{3.14}$$

It now suffices to define a regular and convex φ for which

$$\psi(p) > 2\mathscr{F}(v)4\gamma\widetilde{\psi}(p). \tag{3.15}$$

This can be done since $\widetilde{\psi}$ was defined to be continuous and 0 in a neighborhood of 0. This is per se a trivial fact, although a formal proof of it can be done as follows:

$$\psi(p) = 8\gamma\mathscr{F}(v)\left(\widetilde{\psi}(1)\frac{p^2}{(\frac{1}{16\gamma})^2} + \sum_{n=1}^{+\infty}\widetilde{\psi}(n+1)\left(\frac{p}{n}\right)^{2\sigma_n}\right)$$

where $\sigma_n \in \mathbb{N}$ is a strictly increasing sequence chosen depending on $\psi(n+1)$ to make the radius of convergence of the series infinite. From here one repeats the estimates done in (3.9) bearing in mind (3.7), (3.15), (3.14), and (3.10)

$$\begin{aligned}\mathscr{F}(u) &> \int_0^1 \big(\varphi(u)-\varphi(v)\big)^2\psi(u') > 8\gamma\mathscr{F}(v)\int_a^b \eta\left(\frac{x-x_0}{2\gamma}\right)\widetilde{\psi}(u') > \\ &> 8\gamma\mathscr{F}(v)\int_{(a,b)\setminus H}\frac{u'}{x-x_0} > 8\gamma\mathscr{F}(v)\frac{1}{b-x_0}\int_{(a,b)\setminus H}u' > 2\mathscr{F}(v)\end{aligned}$$

We also deduce from this inequality the for any $u \in \mathrm{Lip}\cap\mathcal{C}$ we have $\mathscr{F}(u)-\mathscr{F}(v) > \mathscr{F}(u)$ so we observe the Lavrentiev gap phenomenon.

We observe that in the case that $u(x_0) > v(x_0)$ the estimates are almost the same but must be carried out on the left side of x_0 i.e. by choosing an appropriate $(b,a) \subset (0,x_0)$.

We have thus proved that E is contained in the singular set of any competing minimizer.

Part 4 Finally we prove in this part that the singular set of u is exactly E and doesn't contain other points.

To begin with, we notice that there is no actual need to provide other proofs since the estimates of part three do not strictly depend on u' being finite in a certain point x_0 where v' is not. The important factor that allows finding $(a,b) \in (x_0,1)$ where the necessary estimates hold is finding a point $x_1 > x_0$ where $u(x_1)-v(x_0) < \frac{1}{4}\big(v(x_1)-v(x_0)\big)$. This effectively proves that for u to be a minimizer u must coincide with v. However therein is given another proof based on Tonelli's results that better illustrates the nature of partial regularity of minimizers.

Reasoning by contradiction consider a point $t_0 \in \overline{I}\setminus E$ where u is singular. We have constructed the functional to satisfy Tonelli's partial regularity theorem 2.1 that gives us the continuity of u' as a function with image in $\mathbb{R} \cup \{\pm\infty\}$. As we have already noticed that u is non-decreasing, we have that $u'(t_0) = +\infty$. Our proof will be based on the analysis of Euler's equation (2.1) that will give the boundedness of u', contradicting our assumption.

Since E is closed, $t_0 \notin E$ and $v'(t_0)$ is finite there is an interval $[t_0-\tau, t_0+\tau]$ on which v' doesn't have singularities. Tonelli's partial regularity theorem gives us that the singular set of u is closed and of zero Lebesgue measure, so for any τ as above it is possible to choose $t_1 \in [t_0-\tau, t_0+\tau]$ such that u is regular in an open neighborhood of t_1. Let's distinguish the cases when $u(t_0) \neq v(t_0)$ and $u(t_0) = v(t_0)$. In the first case the proof is essentially the same for $u(t_0) < v(t_0)$ and $u(t_0) > v(t_0)$ so we will present only one of these two sub-cases.

Writing Euler's equation in integral form starting from the point t_1 we get

$$F_p(t, u(t), u'(t)) = F_p(t_1, u(t_1), u'(t_1)) + \int_{t_1}^{t} F_z(x, u(x), u'(x))\,\mathrm{d}x.$$

In our case this becomes

$$\begin{aligned}\big(\varphi(u) - \varphi(v)\big)^2 \psi'\left(u'\right) + 2u' &= \\ &= c(t_1) + \int_{t_1}^{t} 2\left(\varphi(u) - \varphi(v)\right)\varphi'\left(u\right)\psi\left(u'\right)\,\mathrm{d}x.\end{aligned}$$

where c is a constant depending on t_1. Since $\psi' > 0$ we can write

$$2u' < c(t_1) + \int_{t_1}^{t} 2\left(\varphi(u) - \varphi(v)\right)\varphi'\left(u\right)\psi\left(u'\right)\,\mathrm{d}x \tag{3.16}$$

This inequality must be valid on any interval containing t_1 on which u is regular and so Euler's equation is satisfied.

Case $u(t_0) < v(t_0)$**.** Restrict τ further to have $|\varphi(u) - \varphi(v)| > \varepsilon > 0$. We then have that the integrand of (3.16) is summable on $[t_0 - \tau, t_0 + \tau]$. In fact

$$\begin{aligned}\int_{t_0-\tau}^{t_0+\tau} 2\left|(\varphi(u) - \varphi(v))\varphi'(u)\right|\left|\psi(u')\right|\,\mathrm{d}x &< \\ < K\int_{t_0-\tau}^{t_0+\tau}\left|(\varphi(u) - \varphi(v))^2\varphi'(u)\right|\left|\psi(u')\right|\,\mathrm{d}x &< K\mathscr{F}(u) < +\infty\end{aligned}$$

where $K > 0$ is some constant given by the maximum of the continuous function $\frac{2\left|\varphi'(u)\right|}{\left|\varphi(u)-\varphi(v)\right|}$ over the compact $[t_0 - \tau, t_0 + \tau]$. From (3.16) this implies that u' is uniformly bounded on all intervals containing t_1 on which u is regular.

Case $u(t_0) = v(t_0)$**.** If $t_0 \neq 0$ consider a point t_1, $t_1 < t_0$ and such that for $t \in [t_1, t_0]$ $v(t) \geq u(t)$. This is possible since $v'(t_0) < \infty$ and $u'(t_0) = +\infty$. The integrand in (3.16) is negative or zero so for $t \in [t_1, t_0]$ we have that $u'(t) < c(t_1)$ is uniformly bounded from above for any interval containing t_1 on which u is regular. We also have that $u'(t) > 0$.

Since $u' \to +\infty$ when approaching the boundary of the maximal interval on which u is regular we have that u must be regular at least on an interval containing t_1 and t_0. This contradicts our assumption that $u'(t_0) = +\infty$.

□

4 Tonelli's Partial Regularity Theorem with relaxed regularity assumptions

The version of Tonelli's regularity theorem we have proved in section 2 has a rather unnatural restriction on the regularity of the Lagrangian. As a matter of fact the proof due to Ball and Mizel [1] is applicable only in the settings where the Lagrangian is of class C^3. This is due to the fact that to find a C^1 field of extremals for the application of Hilbert's Theorem 2.5 we need a C^1 dependence on the initial data of the solutions of the Euler equation (2.1). Regardless of the elegance of the proof, this restriction is not natural. As seen in lemma 2.3, C^2 solutions for the Euler equation are defined for C^2 Lagrangians and rewriting the equation in integral form one can speak of C^1 solutions. It is thus reasonable to ask if minimizers of functionals with less regular Lagrangians also possess Tonelli-like regularity. A satisfactory answer to our concerns can be given by the following theorem.

Theorem 4.1 (Tonelli's Partial regularity theorem for functionals with non smooth Lagrangian). *Let $\mathscr{F}$ be a functional with a continuous Lagrangian $F(x, z, p)$ strictly convex in p. Let $F_p(x, z, p)$ exist and be continuous in all $(x, z, p) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$. Finally suppose that F is Lipschitz continuous in z locally in z and p but not necessarily in x, however suppose that the local Lipschitz constant C_R is integrable in x. In other words suppose that for any $R > 0$ there exists $C_R(x) \in L_1(\overline{I})$ such that*

$$|F(x, z, p) - F(x, \widetilde{z}, p)| < C_R(x)|z - \widetilde{z}| \quad \forall |z|, |\widetilde{z}|, |p| < R \tag{4.1}$$

for a.e $x \in \overline{I}$.

Suppose that $w \in AC(I, \mathbb{R})$ is a local AC minimizer for $\mathscr{F}$ in the class $\mathcal{C} = \{u \in AC(I, \mathbb{R}),\ u = w \text{ on } \partial I\}$. Then w has a classical derivative $w' \in C^0(\overline{I}, \mathbb{R} \cup \{\pm\infty\})$. Thus w is regular in the sense of Tonelli.

We will investigate the problem "in small" in a similar manner as in the proof of theorem 2.1 in the regular case. However we will not concentrate on the local solutions to the Euler equation but instead the strict convexity of $F(x, z, p)$ will be used. The reasoning proceeds by absurd around points where regularity of the minimizer fails.

The main idea is to consider Lebesgue points for w' where w' is finite. Let x_0 be such a point and suppose that w' is not continuous there. Then on any neighborhood of x_0 one can find a small interval $\widetilde{I}$ for which the difference quotient of w on the endpoints is very different from $w'(x_0)$. We can then proceed by smoothing out w to bring w' closer to $w'(x_0)$ on $\widetilde{I}$ possibly by substituting w by an affine function with same boundary values as w on some interval containing $\widetilde{I}$. If one operates close enough to x_0 convexity of $F(x, z, p)$ should provide with a decrease in energy of the order of $\mathscr{L}(\widetilde{I})$. Subsequent careful estimates using the Lipschitz assumptions in z on the Lagrangian allow us to prove that other corrections to the energy when smoothing out w are of order inferior to $\mathscr{L}(\widetilde{I})$. Thus the dominant term in the energy variation is given by the convexity in p and so is strictly negative. This is absurd since we have supposed w to be a minimizer.

Such an approach with some additional technicalities actually proves that all points of $\overline{I}$ are Lebesgue points for w', and is continuous when restricted to the set Λ of Lebesgue points for w' where w' is finite. Since $\overline{I} \setminus \Lambda$ has zero Lebesgue measure, Λ is dense. The same convexity argument used to prove continuity on Λ allows for a cotinuous extension of $w'_{|\Lambda}$ to the entire interval $\overline{I}$. A simple interpolation argument will give us that w' on $\overline{I}$ coincides with this continuous extension and thus $w' \in C^0(\overline{I}, \mathbb{R} \cup \{\pm\infty\})$.

We notice that the theorem requires no growth conditions on the Lagrangian and this important detail distinguishes this result from many others in this field. However let's notice that using the continuity assumptions we can assume without loss of generality that locally in (x, z) there is some weak growth condition. In particular we can suppose that for any point (x_0, z_0) there is a neighborhood where

$$F(x, z, p) > |p| \quad \forall p \in \mathbb{R} \tag{4.2}$$

holds. More specifically we have the following lemma.

Lemma 4.2. *Consider a continuous Lagrangian $F(x, z, p) : \overline{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is strictly convex in p. For any $(x_0, z_0) \in \overline{I} \times \mathbb{R}$ there is a neighborhood $A \ni (x_0, z_0)$ and a Lagrangian $F^*(x, z, p)$ such that the functional $\mathscr{F}^*$ associated with F^* has the same local minimizers as the original $\mathscr{F}$ on the domain $\mathcal{C}$ of AC functions with fixed boundary conditions and $F^*(x, z, p)$ satisfies (4.2) for $(x, z) \in A$.*

Proof. We start by noticing that the behavior of $\mathscr{F}$ on $\mathcal{C}$ is invariant under the following transformation of the Lagrangian: $F^*(x, z, p) := \alpha F(x, z, p) + \beta(p - p_0) + \gamma$. This is true since $\int_I \beta(u' - p_0) + \gamma = \mathscr{L}(I)\gamma + \beta(u(b) - u(a) - p_0\mathscr{L}(I))$ where $I = (a, b)$. By applying this transformation we are just adding a constant term and rescaling the functional. Reasoning by continuity and strict convexity of F in p in a sufficiently small neighborhood of (x_0, z_0) we have that it is possible to choose α, β, γ so that (4.2) holds in that neighborhood. □

A particularly important role in the proof of the theorem 4.1 are convexity considerations. We will now prove a lemma that is a weak local version of Jensen's inequality.

Lemma 4.3. *Let $F(y, p)$ be a continuous function convex in p for all y and let $F_p(y, p)$ be continuous; suppose (4.2) holds for all $y = (x, z)$ in a neighborhood of $y_0 = (x_0, z_0)$. Consider $t_0 \in \overline{I}$. Then if $A_n \subset \overline{I}$ is a sequence of measurable sets such that* $\operatorname{diam}(A_n \cup \{t_0\}) \to 0$*, $y \in \mathbb{C}^0(\overline{I})$ and $y(t_0) = y_0$, $\xi \in L_1(I)$ and $⨍_{A_n} \xi(t)\,\mathrm{d}t \to \overline{\xi} \in \mathbb{R}$ then*

$$⨍_{A_n} F(y(t), \xi(t)) \geq F(y(t_0), \overline{\xi}) + o_n(1) \tag{4.3}$$

Proof. Write

$$\begin{aligned} ⨍_{A_n} F(y(t), \xi(t)) &= ⨍_{A_n} F(y(t_0), \overline{\xi}) + ⨍_{A_n} F(y(t), \xi(t)) - F(y(t_0), \overline{\xi}) = \\ &= F(y(t_0), \overline{\xi}) + ⨍_{A_n} F(y(t), \xi(t)) - F(y(t_0), \overline{\xi}) \end{aligned}$$

Now consider the integral on the right-hand side and write

$$
\begin{aligned}
&\fint_{A_n} F(y(t),\xi(t)) - F(y(t_0),\overline{\xi}) = \\
&\fint_{A_n} F(y(t),\xi(t)) - F(y(t),\overline{\xi}) + \fint_{A_n} F(y(t),\overline{\xi}) - F(y(t_0),\overline{\xi}) = \\
&\fint_{A_n} F_p(y(t),\overline{\xi})(\xi(t)-\overline{\xi}) + E(y(t),\overline{\xi},\xi(t)) + \fint_{A_n} F(y(t),\overline{\xi}) - F(y(t_0),\overline{\xi}) \geq \\
&\geq \fint_{A_n} F_p(y(t),\overline{\xi})(\xi(t)-\overline{\xi}) + \fint_{A_n} F(y(t),\overline{\xi}) - F(y(t_0),\overline{\xi}) \geq \\
&\geq \fint_{A_n} F_p(y(t_0),\overline{\xi})(\xi(t)-\overline{\xi}) - \fint_{A_n} \left|F_p(y(t),\overline{\xi}) - F_p(y(t_0),\overline{\xi})\right| \left|\xi(t)-\overline{\xi}\right| + \\
&\qquad + \fint_{A_n} F(y(t),\overline{\xi}) - F(y(t_0),\overline{\xi}),
\end{aligned}
$$

where $E(y,p,q) = F(y,q) - F(y,p) - F_p(y,p,q)$ is the Weierstrass excess function that is always positive by convexity of $F(y,p)$ in p. If $\fint_{A_n} |\xi(t)|$ is bounded then the result follows from the uniform continuity of $y(t)$ and $F(y(t),\overline{\xi})$ in t and from the hypothesis that $\fint_{A_n} \xi(t) - \overline{\xi} \to 0$. However the case when $\fint_{A_n} |\xi(t)|$ is bounded is essentially the only interesting case since for any sub-sequence for which $\fint_{A_n} |\xi(t)| \to \infty$ the estimate (4.3) automatically holds due to (4.2). □

Remark 4.4. *The proof of theorem 4.1 is done "in small". The hypotheses of the theorem provide for w being only a local AC minimizer. However since all reasonings is carried out on small neighborhoods of points of the (x,z) graph of w, the locality of this property is not restrictive. For simplicity's sake we will overlook the estimates necessary to guarantee that we operate on a neighborhood small enough for the minimizing property of w to hold. In any case these are straightforward and require no modification of the proof. We also notice that the proof and thus the theorem does NOT require w to be a strict minimizer.*

Proof of theorem 4.1. We proceed as described initially, with the difference that to allow careful tracking of the derivatives we will have to consider sets that are not necessarily intervals but sometimes are generic measurable sets. This is largely irrelevant to the idea of the proof but is important for some technicalities.

Consider a competing minimizer w and some point $x_0 \in \overline{I}$. In the first part we suppose that that there is a sequence of intervals I_n that get arbitrarily small and close to x_0, on the ends of which the difference quotients of w converge to a limit $p \in \mathbb{R}$. If there were subsets of I_n on which the mean of the derivative w' strayed away from p then w could be slightly modified to lower the value of the functional and this would contradict the minimizing property of w.

We recall that $\Lambda \subset I$ is the set of Lebesgue points for w' such that w' is finite. In the second part we prove that for any fixed $x_0 \in \overline{I}$ the limit $\lim_{x\to x_0} w'(x)$ taken on $x \in \Lambda$ exists. We do this by showing that for any sequence of points $x_n \in \Lambda$ such that $x_n \to x_0$, $w'(x_n)$ conveges to the same value independently of the particular sequence. Reasoning by absurd it is shown that in the case that this doesn't happen, it is possible to choose sets as described in the first part, countrary to the minimality of w.

Initially we show that I_n can be chosen with the needed properties. In general this is true of all points except Lebesgue points of w' such that $|w'(x_0)| = \infty$. For the latter we must use the existence of a sequence $x_n \in \Lambda \to x_0$ such that $w'(x_n) \to \widetilde{p} \neq w'(x_0)$.

Afterwards we choose the subsets of I_n on which the mean of the derivative w' strays away from the difference quotients of w on the border points of I_n. We must distinguish three cases to be able to control where w' is bounded and thus to apply the local Lipschitz condition (4.1). The first, and simpler case, supposes that there exists an interval containing x_0 such that w' is essentially bounded on this interval. We choose a sequence $x_n \in \Lambda \to x_0$ such that setting $\widetilde{p}_n := w'(x_n)$ we have $\widetilde{p}_n \to \widetilde{p} \neq p$. Around these point, we choose the sets where w' strays away from p. Subsequently we deal with the case of unbounded w' near x_0. The sets we will consider are the ones where w' is greater that a certain constant $\overline{\delta}$. A technical distinction is necessary to be able to correctly choose the sets and this accounts for the two remaining cases.

Finally, in the third part, we show how the first two parts imply that w' is continuous on Λ and $w'_{|\Lambda}$ has a continuous extension to the entire interval $\overline{I}$. w' must coincide on $\overline{I}$ with this extension by a simple interpolation argument.

In all of these parts we reason locally around some point $x_0 \in \overline{I}$. Due to lemma 4.2 we will suppose that, at least in some neighborhood of $\big(x_0, w(x_0)\big)$, condition (4.2) holds.

Part 1 Consider a point $x_0 \in \overline{I}$ and suppose that there is a sequence of intervals $I_n = [a_n, b_n] \subset \overline{I}$ such that $\operatorname{diam}\big(I_n \cup \{x_0\}\big) \to 0$ and

$$\frac{w(b_n) - w(a_n)}{b_n - a_n} = \fint_{I_n} w' =: q_n \to p \in \mathbb{R}. \tag{4.4}$$

First we show that we have a convergence in measure of w' to p on I_n in the sense that for any $\delta > 0$

$$\frac{\mathscr{L}\left(\{t \in I_n \mid |w'(t) - p| > \delta\}\right)}{\mathscr{L}(I_n)} \to 0$$

so by a diagonal argument we can find $\delta_n \to 0$ such that

$$\frac{\mathscr{L}\left(\{t \in I_n \mid |w'(t) - p| > \delta_n\}\right)}{\mathscr{L}(I_n)} \to 0 \tag{4.5}$$

holds.

Suppose n large enough for (4.2) to hold and reason by absurd. Define a new competitor v_n such that

$$v'_n(x) = \begin{cases} w'(x) & x \in \overline{I} \setminus I_n \\ p_n & x \in I_n \end{cases}$$

$$v_n(x) = w(0) + \int_0^x v'_n$$

which coincides with w outside I_n. Consider the quantity $\int_{I_n} F(x, w, w') - F(x, v_n, v'_n)$. If $\fint_{I_{n_k}} |w'| \to \infty$ for some subsequence then by (4.2) we have

$$\fint_{I_{n_k}} F(x, w, w') > \fint_{I_{n_k}} |w'| \to \infty$$

while by continuity we have

$$\fint_{I_{n_k}} F(x, v_n, v_n') \to F(x_0, w(x_0), p)$$

so for k sufficiently large we find v_{n_k} such that $\int_{I_{n_k}} F(x, w, w') - F(x, v_{n_k}, v_{n_k}') > 0$ and thus $\mathscr{F}(w) > \mathscr{F}(v_{n_k})$ against the minimizing property of w. On the other hand, if $\fint_{I_n} |w'|$ is bounded, set $E(x, w, p, q) := F(x, w, q) - F(x, w, p) - F_p(x, w, p)(q-p) \geq 0$ to be the Weirstrasse excess function and let's write

$$\begin{aligned}
\int_{I_n} F(x, w, w') - F(x, v_n, v_n') &= \int_{I_n} F(x, w, w') - F(x, w, v_n') + \\
&\quad + \int_{I_n} F(x, w, v_n') - F(x, v_n, v_n') = \\
&= \int_{I_n} F_p(x, w, v_n')(w' - v_n') + \int_{I_n} E(x, w, v_n', w') + \\
&\quad + \int_{I_n} F(x, w, v_n') - F(x, v_n, v_n')
\end{aligned}$$

If there were a $\delta > 0$ such that for some subsequence n_k

$$\frac{\mathscr{L}\left(\{t \in I_{n_k} \mid |w'(t) - p| > \delta\}\right)}{\mathscr{L}(I_{n_k})} > \varepsilon$$

we can rewrite the above using the uniform continuity of $F(x, z, p)$ and the boundedness of $\fint_{I_n} |w' - q_n|$ to have

$$\begin{aligned}
&\int_{I_{n_k}} F(x, w, w') - F(x, v_{n_k}, v_{n_k}') \geq \\
&\qquad \geq -\mathscr{L}(I_{n_k}) \fint_{I_{n_k}} \Big| F_p(x, w, v_{n_k}') - F_p(x_0, w(x_0), p) \Big| \, |w' - v_{n_k}'| + \\
&\qquad +\mathscr{L}(I_{n_k}) \varepsilon \fint_{\{t \in I_{n_k} \mid |w'(t) - p| > \delta\}} E(x, w, v_{n_k}', w') + \\
&\qquad +\mathscr{L}(I_{n_k}) \fint_{I_{n_k}} F(x, w, v_{n_k}') - F(x, v_{n_k}, v_{n_k}') = \\
&\qquad = \mathscr{L}(I_{n_k}) \left(o_{n_k}(1) + \varepsilon \fint_{\{t \in I_{n_k} \mid |w'(t) - p| > \delta\}} E(x, w, v_{n_k}', w') \right)
\end{aligned}$$

noticing that $\int_{I_n} F_p(x_0, w(x_0), p)(w' - v_n') = 0$. For k large enough $|v_{n_k}' - w'| > \frac{\delta}{2}$. Since $F(x, z, p)$ is strictly convex and continuous, $E(x, z, p, q) \geq 0$ is continuous and equal to zero only if $p = q$. It follows that in a neighborhood of $(x_0, w(x_0))$ it is possible to find a non-negative function $c : \mathbb{R} \to \mathbb{R}$ strictly positive outside of 0, such that $E(x, z, p, q) > c(|p - q|) \quad \forall p, q \in \mathbb{R}$ and for all (x, z) in the neighborhood of $(x_0, w(x_0))$. It follows that for k large enough

$$\int_{I_{n_k}} F(x, w, w') - F(x, v_{n_k}, v_{n_k}') \geq \mathscr{L}(I_{n_k}) \left(o_{n_k}(1) + \varepsilon c(\delta/2) \right) > 0.$$

This allows us to find a v_{n_k} such that $\mathscr{F}(w) > \mathscr{F}(v_{n_k})$ in contradiction with the minimizing property of w. Thus (4.5) holds.

From now on all choices of sets are made to be measurable. For each n define $A_n^* = \{t \in I_n \mid |w'(t) - p| < \delta_n\}$ the set where w' is close to p. Suppose there exists $\widetilde{p} \neq p$ and sets $B_n \subset I_n$ such that

$$\widetilde{p}_n := \fint_{B_n} w' \to \widetilde{p} \in \mathbb{R}. \tag{4.6}$$

Finally suppose that $|w'| < \overline{\delta}$ a.e. on $I_n \setminus B_n$ for some $\overline{\delta} > 0$, in other words B_n contains the set where w' is unbounded. For such a choice of B_n we have that

$$\frac{\mathscr{L}(B_n)}{\mathscr{L}(I_n)} \to 0. \tag{4.7}$$

As a matter of fact we can write

$$\frac{\mathscr{L}(B_n)}{\mathscr{L}(I_n)} \fint_{B_n} w' + \frac{\mathscr{L}(I_n \setminus B_n)}{\mathscr{L}(I_n)} \fint_{I_n \setminus B_n} w' = \fint_{I_n} w'.$$

Since w' is bounded on $I_n \setminus B_n$, from the equality above we have that $\frac{\mathscr{L}(B_n)}{\mathscr{L}(I_n)} \leq 1 - \varepsilon$ for some positive ε. Otherwise, passing to the limit over a maximizing subsequence the previous equality would contradict $\widetilde{p} \neq p$. Bearing this in mind (4.5) gives that $\frac{\mathscr{L}(A_n^* \setminus B_n)}{\mathscr{L}(I_n \setminus B_n)} \to 1$. Thus we have

$$\fint_{I_n \setminus B_n} w' = \frac{1}{\mathscr{L}(I_n \setminus B_n)} \left(\int_{A_n^* \setminus B_n} w' + \int_{I_n \setminus (A_n^* \cup B_n)} w' \right) \to p$$

and passing to the limit in the previous equality we have

$$\lim_{n \to \infty} \frac{\mathscr{L}(B_n)}{\mathscr{L}(I_n)} (\widetilde{p} - p) = 0.$$

Since we supposed $\widetilde{p} \neq p$ this gives us the desired result (4.7).

Starting from a certain n we define $A_n \subset A_n^* \setminus B_n$ of the same measure as B_n. This is possible since $\frac{\mathscr{L}(B_n)}{\mathscr{L}(I_n)} \to 0$ and $\frac{\mathscr{L}(A_n^*)}{\mathscr{L}(I_n)} \to 1$. Define $p_n := \fint_{A_n} w'$ and notice that $p_n \to p$. Under these assumptions we will now show that by smoothing out w on A_n and on B_n we exhibit a competitor on which the functional $\mathscr{F}$ has a lower value than on w. Define v_n such that

$$\begin{aligned} v_n'(x) &= \begin{cases} w'(x) & x \in \overline{I} \setminus (A_n \cup B_n) \\ \frac{p_n + \widetilde{p}_n}{2} = \frac{\fint_{A_n} w' + \fint_{B_n} w'}{2} & x \in A_n \cup B_n \end{cases} \\ v_n(x) &= w(0) + \int_0^x v_n' \end{aligned} \tag{4.8}$$

so that w differs from v_n at most on I_n. For this reason we compare the value of the functional for w and for v_n restricted to I_n:

$$\int_{I_n} F(x,w,w') - F(x,v_n,v_n') =$$

$$= \int_{A_n} F(x,w,w') + \int_{B_n} F(x,w,w') - \tag{4.9}$$

$$- \int_{A_n \cup B_n} F(x,v_n,v_n') + \tag{4.10}$$

$$+ \int_{I_n \setminus (A_n \cup B_n)} F(x,w,w') - F(x,v_n,v_n'). \tag{4.11}$$

We now estimate the three terms separately. First of all we apply lemma 4.3 to both terms of (4.9) and then we use the strict convexity of $F(x_0, w(x_0), \cdot)$ bearing in mind $\mathscr{L}(B_n) = \mathscr{L}(A_n)$:

$$\begin{aligned}
&\int_{A_n} F(x,w,w') + \int_{B_n} F(x,w,w') \geq \\
&\quad \geq \mathscr{L}(B_n) F\left(x_0, w(x_0), p_n\right) + \mathscr{L}(B_n) F\left(x_0, w(x_0), \widetilde{p}_n\right) + \mathscr{L}(B_n) o_n(1) \geq \\
&\quad \geq 2\mathscr{L}(B_n) \left(F(x_0, w(x_0), \frac{p_n + \widetilde{p}_n}{2}) + c(|p_n - \widetilde{p}_n|) + o_n(1) \right) + \mathscr{L}(B_n) o_n(1) = \\
&\quad = 2\mathscr{L}(B_n) \left(F(x_0, w(x_0), \frac{p + \widetilde{p}}{2}) + \bar{c}(|p - \widetilde{p}|) + o_n(1) \right) + \mathscr{L}(B_n) o_n(1)
\end{aligned} \tag{4.12}$$

where $c(|p_n - \widetilde{p}_n|)$ is the strict convexity term of $F(x_0, w(x_0), \cdot)$ and is greater than a positive constant $\bar{c}$ dependent only on $p - \widetilde{p}$. Now let's turn to (4.10). Noticing that $v_n' = \frac{p_n + \widetilde{p}_n}{2}$ by uniform continuity of $F(\cdot, v_n(\cdot), \xi)$ for bounded ξ we have

$$\begin{aligned}
\int_{A_n \cup B_n} F(x,v_n,v_n') &= 2\mathscr{L}(B_n) F(x_0, w(x_0), \frac{p_n + \widetilde{p}_n}{2}) + \mathscr{L}(B_n) o_n(1) = \\
&= 2\mathscr{L}(B_n) F(x_0, w(x_0), \frac{p + \widetilde{p}}{2}) + \mathscr{L}(B_n) o_n(1).
\end{aligned} \tag{4.13}$$

Putting together (4.12) and (4.13) we see that the dominating term is $\mathscr{L}(B_n)\bar{c}$. Let us consider (4.11) and prove that its contribution doesn't significantly change the previous estimates. We have supposed that on $I_n \setminus B_n$ w' is bounded uniformly in n so let's say $|w'| < R$. Then, bearing in mind that $w' \equiv v_n'$ on $I_n \setminus (A_n \cup B_n)$, we write

$$\left| \int_{I_n \setminus (A_n \cup B_n)} F(x,w,w') - F(x,v_n,v_n') \right| < \|w - v_n\|_{L_\infty(I_n)} \int_{I_n} C_R(x) \, \mathrm{d}x \tag{4.14}$$

By absolute continuity of the integral $\int_{I_n} C_R(x) = o_n(1)$. We need to estimate $\|w - v_n\|_{L_\infty(I_n)}$. To do this let's use the local growth condition (4.2) we have imposed at the beginning of the proof:

$$\begin{aligned}\|w - v_n\|_{L_\infty(I_n)} &= \sup_{x\in I_n}\left|\int_{x_0}^{x} w' - v_n'\right| \le \int_{A_n\cup B_n} |w'| + \int_{A_n\cup B_n} |v_n'| \\ &\le \int_{A_n\cup B_n} F(x_0, w, w') + 2\mathscr{L}(B_n) \fint_{A_n\cup B_n} \left|\frac{p_n+\widetilde{p}_n}{2}\right|.\end{aligned}$$

Putting these estimates together we have that

$$\left|\int_{I_n\setminus(A_n\cup B_n)} F(x,w,w') - F(x,v_n,v_n')\right| \le o_n(1)\int_{A_n\cup B_n} F(x,w,w') + \mathscr{L}(B_n)o_n(1) \tag{4.15}$$

where the first integral on the right-hand side will be accounted for together with the terms in (4.9).

By using the estimates (4.12), (4.13) and (4.15) in (4.9), (4.10) and (4.11) respectively we get

$$\begin{aligned}&\int_{I_n} F(x,w,w') - F(x,v_n,v_n') \ge \\ &\qquad \ge \big(1+o_n(1)\big)\, 2\mathscr{L}(B_n)\left(F(x,w_0,\frac{p+\widetilde{p}}{2}) + \overline{c} + o_n(1)\right) - \\ &\qquad\qquad -2\mathscr{L}(B_n)F(x,w_0,\frac{p+\widetilde{p}}{2}) + \mathscr{L}(B_n)o_n(1) = \\ &\qquad\qquad = 2\mathscr{L}(B_n)\overline{c} + \mathscr{L}(B_n)o_n(1)\end{aligned}$$

and then for n large enough

$$\int_{I_n} F(x,w,w') - F(x,v_n,v_n') > 0. \tag{4.16}$$

Part 2 Now we show that for any $x_0 \in \overline{I}$ there exists the limit $\lim_{x\to x_0} w'(x)$ taken over $x \in \Lambda$, where Λ is the set of Lebesgue points of w' where w' is finite. We reason by contradiction supposing that there are two sequences $x_n, \widetilde{x}_n \in \Lambda \to x_0$ such that

$$\lim_{n\to\infty} w'(x_n) \neq \lim_{n\to\infty} w'(\widetilde{x}_n) \tag{4.17}$$

and showing that then sequences I_n and B_n can be chosen to satisfy the assumptions of part 1 against the minimizing property of w. For simplicity we deal with the case in which x_0 is not a boundary point of I. The other case is analogous.

Notice that one can always choose some sequence of intervals $I_n = [a_n, b_n]$ with $a_n < x_0 < b_n$ and $a_n, b_n \to x_0$ so that $\fint_{I_n} w' \to p \in \overline{\mathbb{R}}$. If $x_0 \in \Lambda$ then $p \in \mathbb{R}$ is trivially satisfied and one can proceed to choosing the sequence of sets B_n

Suppose now that x_0 is not a Lebesgue point. If $|p| = \infty$ then there must exists an other choice of intervals $\widetilde{I}_n = [c_n, d_n]$ such that $c_n, d_n \to x_0$ and $\fint_{\widetilde{I}_n} w' \to \widetilde{p} \in \overline{\mathbb{R}}$ with $\widetilde{p} \neq p$. Passing to a subsequence we can suppose that $\widetilde{I}_n \subset I_n$. If also $|\widetilde{p}| = \infty$ but

$p \neq \widetilde{p}$ we argue that due to the continuity of $\fint_{[c,d]} w'$ with respect to c and d we can choose c'_n and d'_n such that $[c_n, d_n] \subset [c'_n, d'_n] \subset [a_n, b_n]$ such that $\fint_{[c'_n,d'_n]} w' \to p' \in \mathbb{R}$. Thus at least one of the sequences of intervals among $I_n, \widetilde{I}_n$ and $I'_n = [c'_n, d'_n]$ satisfy the requirements of part 1.

If x_0 is a Lebesgue point for w' with $|w'(x_0)| = \infty$, by our contradiction hypothesis there must exist a sequence $x_n \in \Lambda$ such that $w'(x_n) \to \widetilde{p} \neq w'(x_0)$. Suppose for simplicity that $x_n > x_0$. Since x_n are Lebesgue points by a diagonal argument there exists a sequence τ_n with $|\tau_n| < |x_0 - x_n|$ such that $\fint_{[x_n-\tau_n;x_n+\tau_n]} w' \to \widetilde{p}$. On the other hand $\fint_{[x_0,x_n+\tau_n]} w' \to w'(x_0)$. Reasoning by continuity with respect to the intervals over which we are integrating it is possible to choose a sequence $\widetilde{\tau}_n$ such that $[x_n-\tau_n, x_n+\tau_n] \subset [x_n-\widetilde{\tau}_n, x_n+\tau_n] \subset [x_0, x_n+\tau_n]$ and $\fint_{[x_n-\widetilde{\tau}_n,x_n+\tau_n]} w' \to p \in \mathbb{R}$. Defining $I_n = [x_n - \widetilde{\tau}_n, x_n + \tau_n]$ provides for a sequence of intervals satisfying the requirements of part 1.

We have so far shown that if condition (4.17) holds a sequence I_n of intervals can be chosen to satisfy the conditions of part 1 and we have $p \in \mathbb{R} := \fint_{I_n} w'$. To choose B_n let us consider the cases mentioned in the beginning.

Case 1 Suppose that w' is essentially bounded in some open neighborhood of x_0. Then there exists $\overline{\delta} > 0$ such for n large enough that $|w'| < \overline{\delta}$ a.e. on I_n. Condition (4.17) provides for a sequence $x_n \in \Lambda \cap I_n$ such that $x_n \to x_0$ and and $\xi_n := w'(x_n) \to \xi \neq p$. Since x_n are Lebesgue points for w' for each x_n we can find a $B_n \in I_n$ such that $\fint_{B_n} |w' - \xi_n| < \varepsilon_n$ for some $\varepsilon_n \to 0$. Furthermore since $|w'| < \overline{\delta}$ then $|\xi| < \overline{\delta}$ and $\widetilde{p}_n := \fint_{B_n} w'$ satisfy $|\widetilde{p_n} - \xi_n| < \varepsilon_n$ so $\widetilde{p}_n \to \widetilde{p} := \xi \in R$. The boundedness of w' outside B_n is trivially satisfied.

Case 2 In cases 2 and 3 we suppose that w' is unbounded near x_0. By construction of I_n we have that w' cannot be uniformly bounded on I_n. Start by considering $\overline{\delta} > |p|$ and the respective $B^*_n(\overline{\delta}) = \{x \in I_n \mid |w'(x)| > \overline{\delta}\} \neq \emptyset$. In this sub-case we consider that at least for some $\overline{\delta}$ and some sub-sequence k_n we have $\fint_{B^*_{k_n}(\overline{\delta})} w' \nrightarrow p$. Define $B^*_n := B^*_n(\overline{\delta})$ for such $\overline{\delta}$. Let's choose a sub-sequence k_{m_n} such that $\fint_{B^*_{k_{m_n}}(\overline{\delta})} w' \to \widetilde{\widetilde{p}} \neq p$, possibly infinite. By the regularity of the Lebesgue measure and by the continuity of the mean, it is possible to choose $B_{k_{m_n}} \supset B^*_{k_{n_m}}$ such that $\fint_{B_{k_{m_n}}} w' \to \widetilde{p} \neq p$ with $\widetilde{p}$ finite. Let us pass to the sub-sequence k_{m_n}. The choice we made of B_n satisfies all the conditions of part 1.

Case 3 Suppose that w' is unbounded near x_0 and for any $\overline{\delta}$ and corresponding $B^*_n(\overline{\delta}) = \{x \in I_n \mid |w'(x)| > \overline{\delta}\}$ we have $\fint_{B^*_n(\overline{\delta})} w' \to p$. The procedure adopted in this case is slightly different from the previous cases. In particular we will adopt a small modification of the reasoning of part 1 that leads to the same result. This is due to the fact that is not necessary to choose a set A_n to smooth out w'. In fact the convexity term that we rely on appears when smoothing out only on B^*_n. This is due to the fact that $|w'|$ on B^*_n is very large but the mean converges to a finite limit. First we choose a sufficiently large $\overline{\delta}$ for example $\overline{\delta} := F(x_0, w_0, p) + 1$ and set $B^*_n := B^*_n(\overline{\delta})$. We proceed mimicking the steps we have taken previously. Define

$$\begin{aligned} v_n'(x) &= \begin{cases} w'(x) & x \in \overline{I} \setminus (B_n^*) \\ p_n := \fint_{B_n^*} w' & x \in B_n^* \end{cases} \\ v_n(x) &= w(0) + \int_0^x v_n' \end{aligned}$$

and evaluate

$$\int_{I_n} F(x, w, w') - F(x, v_n, v_n') =$$

$$= \int_{B_n^*} F(x, w, w') - \tag{4.18}$$

$$- \int_{B_n^*} F(x, v_n, v_n') + \tag{4.19}$$

$$+ \int_{I_n \setminus B_n^*} F(x, w, w') - F(x, v_n, v_n'). \tag{4.20}$$

For part (4.18) we will now use the growth condition (4.2) instead of the convexity results of lemma 4.3:

$$\int_{B_n^*} F(x, w, w') > \mathscr{L}(B_n^*)\overline{\delta}.$$

For part (4.19) we estimate using uniform continuity:

$$\begin{aligned} \int_{B_n^*} F(x, v_n, v_n') =& \mathscr{L}(B_n^*) \fint_{B_n^*} F(x, v_n, p_n) = \\ &= \mathscr{L}(B_n^*) F(x_0, w(x_0), p) + \mathscr{L}(B_n^*) o_n(1). \end{aligned}$$

The estimate for the term (4.20) is the same as in (4.14). Estimating $\|w - v_n\|_{L_\infty(I_n)}$ is very similar to the other cases:

$$\begin{aligned} \|w - v_n\|_{L_\infty(I_n)} &= \sup_{x \in I_n} \left| \int_{x_0}^x w' - v_n' \right| \le \int_{B_n^*} |w'| + \int_{B_n^*} |v_n'| \\ &\le \int_{B_n^*} F(x_0, w, w') + \mathscr{L}(B_n^*) \fint_{B_n^*} |p_n|. \end{aligned}$$

So putting everything together and remembering that $\overline{\delta} = F(x_0, w_0, p) + 1$ we have

$$\int_{I_n} F(x, w, w') - F(x, v_n, v_n') >$$

$$\begin{aligned} &> \mathscr{L}(B_n^*) + (1 + o_n(1)) \mathscr{L}(B_n^*) F(x_0, w_0, p) - \\ &- \mathscr{L}(B_n^*) F(x_0, w_0, p) + \mathscr{L}(B_n^*) o_n(1) = \\ &\qquad = \mathscr{L}(B_n^*) + \mathscr{L}(B_n^*) o_n(1) \end{aligned}$$

as previously and so we reach the same conclusion, i.e. that definitely in n

$$\int_{I_n} F(x, w, w') - F(x, v_n, v_n') > 0. \tag{4.21}$$

We have thus proved, reasoning by contradiction, against the minimizing property of w, that for any $x_0 \in \overline{I}$ there exists the limit $\lim_{x \to x_o} w'(x)$ taken over $x \in \Lambda$. In particular this proves that restriction of w' to Λ is continuous.

Part 3 We now show that the continuity of w' on Λ together with the rest of the results of the second part provides us with Tonelli regularity for w on $\overline{I}$. We reason by density of Λ and we use the fact that $\overline{I} \setminus \Lambda$ has zero Lebesgue measure.

First of all let's notice that Λ has full Lebesgue measure. This is true since a.e. point of I is a Lebesgue point of w' and, since w' is integrable, $w' = \pm\infty$ almost nowhere. This also proves that Λ is dense. Since w' is continuous on Λ and for any $x_0 \in \overline{I}$ there exists the limit $\lim_{x \to x_o} w'(x)$ for $x \in \Lambda$ there is a unique extension $\overline{w'}$ of w' onto the metric completion of Λ that by density coincides with $\overline{I}$. $\overline{w'}$ is in the same L_1 class of w' since these two functions coincide on Λ of full measure. Finally we prove that since w' has a continuous representative in L_1 then w is differentiable in all points of $\overline{I}$ and the derivative in every point coincides with the continuous representative. As a matter of fact it suffices to write:

$$\lim_{h \to 0} \frac{w(x+h) - w(x)}{h} = \lim_{h \to 0} \fint_x^{x+h} \overline{w'}(t)\, \mathrm{d}t = \overline{w'}(x)$$

where the last equality is given by the continuity of $\overline{w'}$. This concludes the proof of the theorem.

□

We have thus proved theorem 4.1 in a very reduced regularity setting. In the next section we will see that the integrability condition for the Lipschitz constant in z is necessary. We notice that it is possible to further reduce the regularity requirements for $F(x, z, p)$ by eliminating differentiability in p. The proof of theorem 4.1 without this hypothesis proceeds in a very similar manner. The only part where the differentiability the existence and continuity of $F_p(x, z, p)$ is used is in the proof of the weak version of Jensen's inequality in lemma 4.3 and some similar estimates in the main proof. With some knowledge of non-smooth calculus it is possible to prove the lemma without requiring differentiability of the Lagrangian in p. The idea of the proof is essentially the same using the sub-differential, that, passing to a suitable sub-sequence when needed, plays the same role as the differential in the estimates of lemma 4.3.

Another possible development of this result is permitting regularity assumptions on F to fail on a small subset of the domain. For example one can allow continuity of F to fail in x for some singular set. It is natural to ask that this singular set be closed and of zero Lebesgue measure. For a complete proof of the theorem in such more general settings we recommend the paper of Ferriero the [5]. Here we limit ourselves to stating the complete result:

Theorem 4.5. *Consider a functional*

$$\mathscr{F}(u) = \int_{I\setminus\Sigma} F(x, u(x), u'(x))\,\mathrm{d}x$$

with a continuous Lagrangian, defined on $\mathcal{C} = \{u \in AC(I) \mid u = w \text{ on } \partial I\}$. *Let* $F(x,z,p) : (\overline{I}\setminus\Sigma)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ *be strictly convex in* p *with* $\Sigma\subset\overline{I}$ *closed of zero Lebesgue measure. Suppose that for any* $R>0$ *there exists a function* $C_R(x)\in L_1(\overline{I})$ *such that*

$$\big|F(x,z,p) - F(x,\widetilde{z},p)\big| \le C_R(x)|z-\widetilde{z}|$$

for any $x\in\overline{I}\setminus\Sigma$ *and* $\|z\|, \|z'\|, \|p\| < R$. *Suppose* w *is a local AC minimizer of* $\mathscr{F}$ *on* $\mathcal{C}$. *Then* w *is regular in the sense of Tonelli i.e.* $w'\in C(\overline{I}\setminus\Sigma;\overline{\mathbb{R}^n})$ *and if* $n=1$ *then* $w'\in C(\overline{I}\setminus\Sigma;\overline{\mathbb{R}})$ *where* $\overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty,-\infty\}$.

The proof to theorem 4.1 has been given without mentioning Euler's equations. The regularity conditions on the Lagrangian in the hypothesis of the theorem generally do not allow one to speak of Euler's equations in any sensible way. However, as mentioned at the beginning of this section, and as it can be seen from lemma 2.3, for Lagrangians of class C^2 the Euler equation is well defined. It is a logical question whether a minimizer w has the same regularity of the Lagrangian and if at least locally, where w' is finite, satisfies Euler's equation.

Corollary 4.6. *Suppose we are in the settings of theorem 4.1 and the Lagrangian* $F(x,z,p)\in C^1(A)$ *where* A *an open neighborhood of the* (x,w,w') *graph of* w. *Then on any open interval* $J\subset\overline{I}$ *such that* w' *is finite (also bounded) on* J w *is a solution to Euler's equation in integral form:*

$$F_p(x,w(x),w'(x) = F_p(x_0,w(x_0),w'(x_0)) + \int_{x_0}^{x} F_z(x,w(x),w'(x))\,\mathrm{d}x$$

for any $x, x_0\in J$. *Under* w' *we understand the continuous representative of* w' *in* L_1. *Furthermore if* $F(x,z,p)\in C^2(A)$ *then* $w\in C^2(J)$ *and*

Proof. The proof is classical. For a reference see [2] and the Appendix. □

5 A singular minimizer for a functional with continuous Lagrangian

After having significantly lowered the necessary regularity conditions on the Lagrangian for applying Tonelli's Partial Regularity Theorem we now show that continuity alone of $F(x,z,p)$ even in all the three variables is not a sufficient condition for Tonelli-like regularity of minimizers. We will exhibit a series of examples of functionals that have continuous Lagrangians but possess minimizer that have discontinuous or undefined derivatives. In the end we will see that the integrability of the Lipschitz constant imposed in the conditions of theorem 4.1 cannot be droped since it is the only condition that fails in the examples we will exhibit.

To better describe the singularities of the minimizers let us introduce the following object:

Definition 5.1. *For a function u we define the upper and lower Dini derivatives as:*

$$\overline{D}u(x_0) = \limsup_{x \to x_0} \frac{u(x) - u(x_0)}{x - x_0}$$
$$\underline{D}u(x_0) = \liminf_{x \to x_0} \frac{u(x) - u(x_0)}{x - x_0}.$$

In our examples we will construct functionals

$$\mathscr{F}(u) = \int_{-T}^{+T} F(x, u, u')\,\mathrm{d}x$$

with continuous strictly convex Lagrangians whose minimizers among AC functions with appropriate boundary values are Lipschitz-continuous but do not possess regularity in the sense of Tonelli. In particular we will exhibit functionals whose minimizers have different upper and lower Dini derivatives in some points and thus Tonelli regularity fails there since u' doesn't exist. In the first example this behavior occurs in a boundary point of I. Subsequently we show that the same thing can happen in an interior point of I, and also in any finite number of points of I. We will also give the main ideas used to construct a functional whose minimizer fails to be Tonelli-regular on a countable dense set. The complete details and technicalities to construct the functional are rather complicated and the interested reader is invited to read the article by Gratwick and Preiss [6].

First, to illustrate the starting point for the constructions of the counterexamples to the Tonelli theorem, we begin with a simple model case which shows, given a fixed smooth function w defined on an interval I, how to build a continuous functional that has w as the minimizer in $\mathcal{C} = \{u \in AC(I) \mid u = w \text{ on } \partial I\}$.

Example 5.2. *Consider the Lagrangian*

$$F(x, z, p) = \varphi(x, z - w(x)) + p^2 \tag{5.1}$$

with

$$\varphi(x, v) = 2|v|(1 + w''(x)^2)^{1/2}.$$

Then the associated functional $\mathscr{F}$ has w as its minimizer.

Proof. Consider a competitor u and write

$$\mathscr{F}(u) - \mathscr{F}(w) = \int_I \varphi(x, u - w) + u'^2 - w'^2 \geq \int_I \varphi(x, u - w) + 2(u' - w')w'.$$

Now integrate by parts. The boundary terms will vanish since u and w coincide on the extremal points of I:

$$\int_I \varphi(x, u - w) + 2(u' - w')w' = \int_I \varphi(x, u - w) - 2w''(x)(u - w) \geq$$
$$\geq \int_I \varphi(x, u - w) - 2|w''(x)||u - w|.$$

Substituting the definition of φ we have

$$\mathscr{F}(u) - \mathscr{F}(w) \geq 2 \int_I |u-w|(1+w''(x)^2)^{1/2} - |w''(x)||u-w|.$$

The integrand is positive so we have $\mathscr{F}(u) - \mathscr{F}(w) \geq 0$ where the inequality is strict if $u \neq w$ as required. $\square$

We will now construct a functional with Lagrangians of the same general form as (5.1) where we initially choose the target minimizer w so that it doesn't possess regularity in the sense of Tonelli. This procedure, as can be guessed from the above example, will require careful tracking of the derivatives of w (where they exists) up to the second order. As always, the class $\mathcal{C}$ of competing functions will be $\{u \in AC(I) \mid u = w \text{ on } \partial I\}$ where I is the interval on which we will consider functional. The continuous penalty function $\varphi(x, v)$ will be chosen accordingly so as to sufficiently penalize competitors not coinciding with the target AC minimizer w, and thus to make w the minimizer. $\varphi(x, v)$ will be non-negative and non-decreasing in $|v| := |u - w|$. As usual $\varphi(x, 0) = 0$.

In the first example we will choose w with one point of non differentiability in 0 and the interval will be $I = [0, T]$. Since any competing minimizer is forced to pass through the singularity of w in 0 we will show that regularity in the sense of Tonelli can fail on the boundary points.

In the second example will consider the same target minimizer w but we set the interval to be $I = [-T, T]$. Adding another term to the penalty function we will exhibit a case where regularity of the minimizer fails for an interior point.

In our examples we will rely on the existence of functions w that are continuous on $[-T, T]$ and smooth outside 0. In zero we require the classical derivative u' not to exist and in particular we ask that that both the right and left Dini derivatives are $+1$ and -1 rispectively: $\overline{D}^{+,-}u(0) = +1$ and $\underline{D}^{+,-}u(0) = -1$. For our example we will choose a base function w that satisfies $-|x| \leq w(x) \leq |x|$ and in a neighborhood of 0, being continuous, oscillates infinitely many times between the two bounding functions $\pm|x|$.

5.1 Oscillating singular functions

Let us consider the class of functions with such behavior in a little more detail. A standard example is

$$w(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where its value in 0 is defined by continuity. This function respects the condition $-|x| \leq w(x) \leq +|x|$. Furthermore for the sequence $x_k = \frac{1}{k\pi + \pi/2} \to 0$ we have that $w(x_{2k}) = 1$ and $w(x_{2k+1}) = -1$. These two facts provide for the conditions imposed on the Dini derivatives. Unfortunately this function is neither Lipschitz continuous nor AC which makes it unsuitable for our needs. However we can generalize this example by setting

$$w(x) = \begin{cases} x \sin\left(h\left(\frac{1}{x}\right)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

with $h : \mathbb{R} \to \mathbb{R}$ a smooth increasing function such that $\lim_{t\to\infty} h(t) = \infty$. For the sake of our subsequent examples we will consider $h(-x) = h(x)$ i.e. h to be an even function. If we choose h to be increasing very slowly we can make w oscillate very slowly and thus be Lipschitz. As a matter of fact we check for $x \neq 0$ that we have

$$w'(x) = \sin\left(h(\frac{1}{x})\right) - \frac{h'\left(\frac{1}{x}\right)}{x}\cos\left(h(\frac{1}{x})\right),$$

so

$$|w'(x)| < 1 + \frac{h'\left(\frac{1}{x}\right)}{x}.$$

Since $\frac{1}{t} \notin L_1(\mathbb{R}^+)$ it is still possible to choose $h'(t)t < 1$ or even $h'(t)t = o(1)$ for $t \to \infty$ and still $\lim_{t\to\infty} h(t) = \infty$. All we need is to be sure that $h'(t)$ is not summable towards $+\infty$. For such a choice of h, by the Lagrange theorem we have that w is Lipschitz. As stated earlier, it will be necessary to keep track of the second derivative of w. Obviously w'' cannot be bounded since w' oscillates infinitely many times any small neighborhood of 0. However calculating w'' for $x \neq 0$ we have

$$w''(x) = \frac{h''\left(\frac{1}{x}\right)\cos\left(h\left(\frac{1}{x}\right)\right) - h'^2\left(\frac{1}{x}\right)\sin\left(h\left(\frac{1}{x}\right)\right)}{x^3}.$$

Simple calculations allow us to see that it is possible to choose h satisfying the initial hypothesis so that $h''(t)t^2 = o(1)$ and $h'(t)t = o(1)$ for $t \to \infty$. This gives us that $xw''(x) = o(1)$ for $x \to 0$. We thus summarize the important properties of w:

$$\begin{aligned} |w(x)| &< |x| \\ \mathrm{Lip}\,(w) &< 2 \\ |w'(x)| < 1 + o(1),\ x|w''(x)| &= o(1) \quad \text{as } x \to 0 \\ \overline{D}^{+,-}w(0) &= +1 \\ \underline{D}^{+,-}w(0) &= -1. \end{aligned} \tag{5.2}$$

We also have

$$|h'(t)| = o(t^{-1}) \quad \text{as } t \to \infty. \tag{5.3}$$

In the following examples we will show where these choices are relevant. As an example of w and h that satisfy these conditions we will take $h(t) = \log\log\log(|t|)$ so that

$$w(x) = x\sin\left(\log\log\log\left(\frac{1}{|x|}\right)\right). \tag{5.4}$$

The above-mentioned properties can be checked by means of simple calculations. In all further examples we will use $w(x)$ as a base function. Furthermore for the choice of w as in (5.4) consider $T = \frac{e^{-e}}{2}$.

5.2 The first example: a singularity in a boundary point

Example 5.3 (Singularity in a boundary point)**.** *Consider the function $w(x)$ as above and the functional $\mathscr{F}(u) = \int_0^T \Big(\varphi\left(x, u(x) - w(x)\right) + u'^2(x)\Big)\,\mathrm{d}x$ defined over the class of competing functions $\mathcal{C} = \{u \in AC([0,T]) \mid u = w \text{ on } \partial I\}$. Set*

$$\varphi(x,v) = \begin{cases} 15|x|w''(|x|) & \text{if } |v| > 5|x| > 0 \\ 3|v|w''(|x|) & \text{if } |v| \le 5|x| \ne 0 \; . \\ 0 & \text{if } x = 0 \end{cases} \tag{5.5}$$

Then $\varphi(x,v)$ and the Lagrangian are continuous and w is a strict minimizer of $\mathscr{F}$ over $\mathcal{C}$. Tonelli-like regularity for the minimizer w fails in the boundary point $x = 0$

Proof. We must prove that for any $u \in \mathcal{C}$ different from w we have that $\mathscr{F}(u) > \mathscr{F}(w)$. Obviously, if $\mathscr{F}(u) = \infty$ then there is nothing to prove. We can thus consider $u' \in L_2$. w is Lipschitz so no problems with convergence of integrals will arise. Bearing in mind that $\varphi(x,0) = 0$ for any x, consider

$$\mathscr{F}(u) - \mathscr{F}(w) = \int_0^T \varphi(x, |u-w|) + \int_0^T u'^2 - w'^2. \tag{5.6}$$

We will now prove this quantity to be positive by first considering an estimate for the term with the derivatives and then showing that a suitable penalty function can be found. The delicate point is that since φ is continuous and is small when when $|u - w|$ is small, for any reasonable minimizer u, the penalty function tends to zero in a neighborhood of $x = 0$. This means that the integral of the penalty function over a τ neighborhood of $x = 0$ is of order $o(\tau)$ so care should be take to make sufficiently accurate estimates of the integral of the derivative terms.

Part 1 Here we deal with the derivative terms. First we write

$$\int_0^T u'^2 - w'^2 > \int_0^T 2(u' - w')w'$$

and integrate by parts. Care must be taken since w' isn't defined in 0. We integrate by parts on $[\varepsilon, T]$:

$$\int_\varepsilon^T 2(u'-w')w' = 2\left[(u-w)w'\right]_\varepsilon^T - \int_\varepsilon^T 2(u-w)w'' > 2\left[(u-w)w'\right]_\varepsilon^T - \int_\varepsilon^T 2|u-w||w''| \tag{5.7}$$

Then we take the limit for $\varepsilon \to 0$. Since w' is bounded, $|u(x) - w(x)| \to 0$ as $x \to 0$ and the left boundary term vanishes. The right boundary term is zero because of the boundary conditions imposed on the competing functions. Let us suppose for now that the competing minimizer u also satisfies $|u| < |x|$ while $|w| < |x|$ by (5.3). A reason for which this is always true will be given below. Then $|u - w| < 2x$ and since by (5.2) $xw''(x) = o(|1|)$ the last integral of the chain of inequalities converges on $[0, T]$. Passing to the limit we can thus write:

$$\int_0^T u'^2 - w'^2 > -2\int_0^T |u-w||w''|. \tag{5.8}$$

Part 2 We will now adequately choose the penalty function φ. It suffices to set $\varphi(x,|u-w|) = 3|u-w||w''(x)|$. Supposing as before $|u| < |x|$, from (5.6) and (5.8) we get

$$\begin{aligned}\mathscr{F}(u) - \mathscr{F}(w) &= \int_0^T \varphi(x,|u-w|) + \int_0^T u'^2 - w'^2 > \\ &> \int_0^T \varphi(x,|u-w|) - 2|u-w||w''| \geq \int_0^T 3|u-w||w''| - 2|u-w||w''| \geq 0\end{aligned} \tag{5.9}$$

where inequality is strict if $u \neq w$. This means that w is a strict minimizer.

Part 3 To prove that if u is a minimizer then $|u| < |x|$ we argue by convexity. If $|u| > |x|$ in some point then by continuity the inequality would be true on an interval (α, β). Consider u^* given by substituting u by an affine function on the interval $[\alpha, \beta]$ connecting points $(\alpha; u(\alpha))$ and $(\beta; u(\beta))$. Now comparing $\mathscr{F}(u^*)$ and $\mathscr{F}(u)$ we get

$$\int_\alpha^\beta u^{*\prime 2} - u'^2 < 0$$

$$\int_\alpha^\beta \varphi(x,|u^*-w|) - \varphi(x,|u-w|) < 0.$$

The first is due to the convexity of p^2 the integral of which notably assumes minimum for affine functions (strait lines) among functions with same boundary values. The second is due to the monotonicity of $\varphi(x,v)$ in $|v|$.

Part 4 Finally we notice that $\varphi(x,v)$ defined as above is continuous for $x \neq 0$ because w is C^∞ outside $x = 0$. Since we have proved that for any minimizer $|u-w| < 2|x|$ we can redefine $\varphi(x,v)$ as needed for $|v| > 2|x|$ taking care to maintain monotonicity in $|v|$ and continuity. Considering that $|x|w''(|x|) \to 0$ as $x \to 0$ given by (5.2), we redefine $\varphi(x,v)$ for $|v| > 5|x|$ as in (5.5). Thus defined, $\varphi(x,v)$ is continuous. In fact in $(0;v)$ we have $\varphi(0,v) = 0$ and considering sequences $x_n \to 0$ and $v_n \to v$ we have $3|v_n|w''(|x_n|) < 15|x_n|w''(|x_n|) \to 0$ as needed.

□

Remark 5.4. *The considerations we made to cope with the difficulty of integrating by parts to obtain (5.8) is actually central to the proof. w' is not defined in 0 and w'' is not bounded. To pass to the limit on the boundary terms we used the boundedness of w' equivalent to w being Lipschitz together with the fact that $u - w \to 0$ as $x \to 0$. However, in more general settings like those of the subsequent examples, we will have to use a different approach as there is no a priori reason for u and w to coincide in the point of singularity.*

The second estimate we made in (5.7) relative to the term with w'' depends heavily on the growth conditions of w'' established by (5.2). If w'' were strictly of order $\frac{1}{x}$ or greater as $x \to 0$ passing to the limit could be impossible and in any case defining φ as in (5.5) wouldn't yield a continuous functional.

Actually the conditions that we have imposed on w'' are not specific to the estimates we chose to make in our proof. Heuristically, controlling w'' or equivalently controlling *the fact that w doesn't oscillate too quickly is necessary to be able to force a candidate minimizer to coincide with w. If it were to oscillate too quickly it becomes advantageous energy-wise to substitute w with constant 0 in a neighborhood of the singularity.*

5.3 The second example: singularity in an interior point

Before proceeding to the second example we first prove a lemma about the behavior of candidate minimizers in the neighborhood of points where they differ from the target minimizer.

Lemma 5.5. *Consider a Lagrangian as in (5.1) with w a target minimizer with Lipschitz constant L. Consider the domain $\mathcal{C} = \{u \in AC(I) \mid u = w \text{ on } \partial I\}$ of competitor functions with fixed boundary values. Let $u \in \mathcal{C}$ a minimizer of the associated functional and suppose x_0 is a point where $u(x_0) \neq w(x_0)$. Then there is an open interval J containing x_0 where u is Lipschitz continuous with the constant L. If $u(x_0) > w(x_0)$ then u is convex on that interval while if $u(x_0) < w(x_0)$ then u is concave. Furthermore the following limits on the measure of J hold:*

$$\frac{2}{2L+1}\left|u(x_0) - w(x_0)\right| \leq \mathscr{L}(J) \leq 2\left|u(x_0) - w(x_0)\right|. \tag{5.10}$$

Finally

$$\left|u(x) - w(x)\right| > \frac{1}{2L+1}\left|u(x_0) - w(x_0)\right| \geq \frac{1}{2(2L+1)}\mathscr{L}(J) \quad \textit{for } x \in J \tag{5.11}$$

Proof. We prove the lemma when $u(x_0) > w(x_0)$. The other case is symmetric. Consider the maximal open interval J^* with $x_0 \in J^*$ such that

$$u(x) - w(x_0) > L|x - x_0| \quad \text{for } x \in J^*$$

holds. This interval is a subset of I since for the endpoints we have $u(x) - w(x_0) = w(x) - w(x_0) \leq L|x - x_0|$ by Lipschitzianity of w. We will now prove that u is convex on J^*. As a matter of fact let $\alpha, \beta \in J$ be two points such that the affine function l connecting $(\alpha; u(\alpha))$ and $(\beta; u(\beta))$ lies underneath the graph of u i.e. $l(x) \leq u(x)$ for $x \in (\alpha, \beta)$. This contradicts the minimality of u since by substituting u with l on (α, β) we obtain a function u^* for which $\varphi(x, u^*) \leq \varphi(x, u)$ since $|w - u^*| < |w - u|$ by construction. We thus have

$$\int_\alpha^\beta \varphi(x, u^*) \leq \int_\alpha^\beta \varphi(x, u)$$

and

$$\int_\alpha^\beta u^{*\prime 2} \leq \int_\alpha^\beta u'^2$$

by convexity of p^2. The second inequality is strict if u and u^* don't coincide. If u and u^* didn't coincide this would contradict the minimality of u.

We have that u is Lipschitz with constant L on J^*. In fact if there were $\alpha, \beta \in J^*$ such that $\left|\frac{u(\beta)-u(\alpha)}{\beta-\alpha}\right| > L$ then, due to convexity on J^*, u couldn't intersect the graph of $z = L|x - x_0| + w(x_0)$ both to the left and to the right of x_0. But that must happen for the same reason as above, i.e. since w is L-Lipschitz continuous and w and u coincide on the boundary points of I.

Now let $J \subset J^*$ be the maximal interval in J^* such that

$$u(x) - w(x_0) > (L+1)|x - x_0| \quad \text{for } x \in J.$$

All the properties proved for J^* also hold on J. We can estimate the length of J. Suppose $J = (a, b)$ for some $a, b \in I$. We have that $u(b)-w(x_0) = (L+1)(b-x_0)$ so by the Lipschitz property of u proven above we have that $b - x_0 \leq u(x_0) - w(x_0) \leq (2L+1)(b - x_0)$. The same applies to a and we have (5.10):

$$\begin{aligned} b - a &\leq 2(u(x_0) - w(x_0)) \\ b - a &\geq \frac{2}{2L+1}(u(x_0) - w(x_0)). \end{aligned} \tag{5.12}$$

Finally we prove estimate (5.11). Using the L-Lipschitz continuity of u and w we have that $|u(x) - w(x)| \geq |u(x_0) - w(x_0)| - 2L|x - x_0|$ while by definition of J we have that $|u(x) - w(x)| \geq |x - x_0|$. The first inequality of (5.11) follows. Using (5.10), that has been already proved, we have the second inequality and thus the result. □

Corollary 5.6. *As a matter of fact this lemma proves that any competitor minimizer from $\mathcal{C}$ is Lipschitz continuous with constant L on all I. It suffices to apply the lemma on any point in $I \setminus \{x \in I \mid u(x) = w(x)\}$. This set is open and the lemma gives local Lipschitz continuity for u with constant L that extends trivially to the entire interval I.*

We now proceed to the second example. We suppose w to be the same as in example 5.3.

Example 5.7 (Singularity in an interior point)**.** *Consider the function $w(x)$ and the functional $\mathscr{F}(u) = \int_{-T}^{T} \Big(\varphi\left(x, u(x) - w(x)\right) + u'^2(x)\Big)\,\mathrm{d}x$ defined over the class of competing functions $\mathcal{C} = \{u \in AC([-T, T]) \mid u = w \text{ on } \partial I\}$. Set $\varphi^{(1)}(x, v)$ to be the same continuous penalty function used in example 5.3. Then there exists a continuous function $\varphi^{(2)}(x, v)$ that satisfies all the requirements for a penalty function. Setting $\varphi = \varphi^{(1)} + \varphi^{(2)}$ gives us a penalty function that makes w a strict minimizer of $\mathscr{F}$ over $\mathcal{C}$. Tonelli-like regularity for the minimizer w fails in the interior point $x = 0$.*

An example of such a $\varphi^{(2)}$ can be

$$\varphi^{(2)}\left(x, v\right) = \sup_{0<\delta\leq 2(2L+1)v} 4\frac{|h(1/\delta)|}{\delta} + \sup_{0<\delta\leq 2(2L+1)v} \frac{16}{\delta}\int_0^\delta \left|\frac{w(\delta)}{\delta} - w'(s)\right| \mathrm{d}s.$$

Thus defined $\varphi^{(2)}$ is continuous and satisfies all the requirements for a penalty function.

Proof. Our proof goes as follows. We consider a competing minimizer $u \in \mathcal{C}$ and we distinguish two cases. The first, simple, case is when $u(0) = w(0) = 0$ that is equivalent to the case in example 5.3. As a matter of fact, since the penalty functions $\varphi^{(2)}$ is non-negative and zero if $u = w$, the same procedure as in example 5.3 symmetrically to $[0; T]$ and to $[-T; 0]$ yields the result.

There is no a priori reason for u to pass through the singular point of w. From now on let's consider the case $u(0) \neq w(0) = 0$. We first compare the integrals of the

Lagrangians, for u and for w, on an open interval J containing 0 (given by lemma 5.5) and then we compare the integrals outside J.

On J we will do our estimates using the other penalty function $\varphi^{(2)}$. By using (5.10) we will show that for however large a function $\eta(d)$ that satisfies $\eta(d) = o(d)$ for $d \to 0$ an appropriate penalty function $\varphi^{(2)}$ can be chosen so that the contribution of the penalty corresponding to $\varphi^{(2)}$ over J is greater than $\eta(\mathscr{L}(J))$.

Bearing in mind this result, we will try to prove that the contribution over J of the integral of the derivative terms of u differs from that of w by a quantity that goes to zero more than linearly in $\mathscr{L}(J)$. This would permit us to choose an appropriate $\varphi^{(2)}$ to penalize u sufficiently on J. We cannot integrate by parts directly as done previously since w is not regular on J. Instead of comparing directly with w we compare u with an affine function l having the same boundary values as w on J. We will prove that the error made by approximating w with such an affine function is sufficiently small.

Subsequently, integrating by parts on J the difference $u'^2 - l'^2$, l'' vanishes so only the boundary terms remain. The procedure outside J is essentially the same as in example 5.3, and the relevant penalty function is $\varphi^{(1)}$, with the difference that the boundary terms obtained after integrating by parts do not vanish but have to be accounted for. We will prove that the difference of the boundary terms given by the two integrations by parts, one inside J and the other one outside, almost cancel out, up to an acceptable order of $o(\mathscr{L}(J))$.

Finally we will gather the estimates made so far and find a suitable $\varphi^{(2)}$

Part 1 In this part we give an estimate for the integral of the penalty function over J

As stated above, the proof for the case $u(0) = w(0)$ is identical to the one in example 5.3. In fact it suffices to consider the problem separately on $[0, T]$ and symmetrically on $[-T, 0]$. If $u(0) \neq w(0)$ we can suppose that $u(0) > w(0)$; in the other case the proof is identical. By applying lemma 5.5 in $x_0 = 0$, whose hypothesis are satisfied because of (5.2), we obtain $J = (-a, b)$. Suppose that $d = \max(a, b)$ and for simplicity of notation suppose $d = b \geq a$, while let $L = 2$ as per (5.2) be the Lipschitz constant for w. The penalty function $\varphi^{(2)}(x, v)$ does not depend on x so from now on we will omit x and for the ease of notation write $\varphi^{(2)}(v) := \varphi^{(2)}(x, v)$. For any penalty function $\varphi^{(2)}$ we have by inequality (5.11) from the lemma:

$$\int_J \varphi^{(2)}(u - w) > \int_0^d \varphi^{(2)}(u(x) - w(x)) >$$
$$> \varphi^{(2)}\left(\frac{1}{2(2L+1)}\mathscr{L}(J)\right) d \geq \varphi^{(2)}\left(\frac{d}{2(2L+1)}\right) d.$$

Then for any non-negative $\eta(d)$ satisfying $\eta(d) = o(d)$ for $d \to 0$ choose

$$\varphi^{(2)}\left(\frac{d}{2(2L+1)}\right) = \sup_{0<\delta\leq d} \frac{\eta(\delta)}{\delta}. \tag{5.13}$$

Notice that such a definition provides us with a continuous $\varphi^{(2)}$ if η is continuous. With the previous inequality we have

$$\int_J \varphi^{(2)}(u - w) \geq \eta(d). \tag{5.14}$$

Part 2 Here we approximate w with an affine function $l(x)$ and estimate the difference of the integral of the derivative term of the Lagrangian for u and for l over the interval where l replaces w.

With the assumptions of part 1 let us extend J and let us consider a symmetric interval $\widetilde{J} = (-d, d) \supset J$. Consider the affine function l on $\widetilde{J}$ passing through $(-d; w(-d)$ and $(d; w(d))$. Since w is odd we have $l' = \frac{w(d)}{d}$. Let us estimate the difference in the integrals of the derivative terms by substituting l to w. Integrating by parts we have

$$\int_{-d}^{d} u'^2 - l'^2 \geq \int_{-d}^{d} 2l'(u' - l') = 2l'[(u-l)]_{-d}^{d} = 2l'[(u-w)]_{-d}^{d}. \tag{5.15}$$

For now we name these terms

$$\begin{aligned} \mathcal{I}^+(d) &= \big(2l'(u-w)\big)(d) \\ \mathcal{I}^-(d) &= \big(2l'(u-w)\big)(-d) \end{aligned} \tag{5.16}$$

and we will later compare them with the boundary terms of integrating by parts outside J.

Part 3 Here we prove that the error made by substituting w by l on $\widetilde{J}$ is sufficiently small. From (5.2) we have the Lipschitzianity of w and thus of l so we write:

$$\begin{aligned} \left| \int_{\widetilde{J}} l'^2 - w'^2 \right| &\leq \max_{\widetilde{J}} \big(|l' + w'|\big) \int_{-d}^{d} |l' - w'| < 4 \int_{-d}^{d} |l' - w'| = \\ &= 4 \int_{-d}^{d} \Big|\frac{w(d)}{d} - w'(s)\Big| \,\mathrm{d}s = 8 \int_{0}^{d} \Big|\frac{w(d)}{d} - w'(s)\Big| \,\mathrm{d}s. \end{aligned} \tag{5.17}$$

A careful estimate of the last integrand is necessary. Let's use the Lagrange theorem to write

$$\begin{aligned} \left|\frac{w(d)}{d} - w'(s)\right| &= \left|\frac{w(d)}{d} - \frac{w(s)}{s} + \cos\big(h(1/s)\big)\,\frac{h'(1/s)}{s}\right| \leq \\ \leq \left|\frac{w(d)}{d} - \frac{w(s)}{s}\right| + \left|\frac{h'(1/s)}{s}\right| &= |d-s| \left|\cos\big(h(1/\theta_s)\big)\,\frac{h'(1/\theta_s)}{\theta_s^2}\right| + \left|\frac{h'(1/s)}{s}\right| \end{aligned}$$

for some intermediate point $\theta_s \in (s, d)$. The previous expression is bounded, but this is not enough as we want the integral to be of order $o(d)$. Now we choose to separate the integral in to two regions: $[0, \delta]$ with δ very close to zero as to have negligible measure and $(\delta, d]$ where a more careful estimate of the integrand is possible. Bearing in mind that w is Lipschitz continuous, we write considering the order in d:

$$\begin{aligned} \int_0^d \left|\frac{w(d)}{d} - w'(s)\right| \,\mathrm{d}s &\leq 4\delta + \int_\delta^d \left(|d-s| \left|\cos\big(h(1/\theta_s)\big)\,\frac{h'(1/\theta_s)}{\theta_s^2}\right| + \left|\frac{h'(1/s)}{s}\right| \right) \mathrm{d}s \leq \\ &\leq 4\delta + d\frac{d}{\delta} \sup_{\theta\in[\delta,d]} \left|\frac{h'(1/\theta)}{\theta}\right| + o(d) \leq 4\delta + d\frac{d}{\delta} \sup_{\theta\in(0,d]} \left|\frac{h'(1/\theta)}{\theta}\right| + o(d) \end{aligned} \tag{5.18}$$

Now it is possible to choose $\delta = d\left(\sup_{\theta\in(0,d]}\left|\frac{h'(1/\theta)}{\theta}\right|\right)^{1/2}$ and since $\sup_{\theta\in(0,d]}\left|\frac{h'(1/\theta)}{\theta}\right| = o(1)$ by (5.3) we have the necessary estimate for (5.17):

$$\left|\int_{\widetilde{J}} l'^2 - w'^2\right| \le 8\int_0^d \left|\frac{w(d)}{d} - w'(s)\right| \mathrm{d}s = o(d). \tag{5.19}$$

Part 3 Here let's estimate the behavior of the integrals outside $\widetilde{J}$ and then account for the boundary terms (5.16) from inside $\widetilde{J}$. As done example 5.3 integrating by parts on $I \setminus \widetilde{J}$ gives

$$\int_{I\setminus\widetilde{J}} u'^2 - w'^2 \ge \int_{I\setminus\widetilde{J}} 2(u'-w')w' =$$
$$= 2\left[(u-w)w'\right]_{-T}^{-d} + 2\left[(u-w)w'\right]_d^T - 2\int_{I\setminus\widetilde{J}}(u-w)w''.$$

Apart from the boundary terms

$$\begin{aligned}\mathcal{E}^+(d) &= \big(2w'(u-w)\big)(d)\\ \mathcal{E}^-(d) &= \big(2w'(u-w)\big)(-d)\end{aligned} \tag{5.20}$$

exactly as in (5.8) we have

$$\int_{I\setminus\widetilde{J}} u'^2 - w'^2 \ge -\mathcal{E}^+(d) + \mathcal{E}^-(d) - 2\int_{I\setminus\widetilde{J}} |u-w||w''|$$

and exactly as in (5.9) we have

$$\int_{I\setminus\widetilde{J}}\left(\varphi^{(1)}(x,u-w) + u'^2 - w'^2\right) \ge -\mathcal{E}^+(d) + \mathcal{E}^-(d). \tag{5.21}$$

Finally we estimate

$$\mathcal{I}^+(d) - \mathcal{E}^+(d) - \mathcal{I}^-(d) + \mathcal{E}^-(d) \le \left|\mathcal{I}^+(d) - \mathcal{E}^+(d)\right| + \left|\mathcal{I}^-(d) - \mathcal{E}^-(d)\right|.$$

For the first term, using the Lipschitzianity of u and w, we have $|u-w| < 4d$ and so

$$\left|\mathcal{I}^+(d) - \mathcal{E}^+(d)\right| = \left|2(l'-w')(u-w)\right|(d) \le 8d\left|\cos\left(h(1/d)\right)\frac{h(1/d)}{d}\right|$$

evaluating the derivative terms explicitly. We know from (5.3) that $\frac{h(1/d)}{d} = o(1)$. Making the equivalent estimates for $\left|\mathcal{I}^-(d) - \mathcal{E}^-(d)\right|$ we finally deduce

$$\left|\mathcal{I}^+(d) - \mathcal{E}^+(d)\right| + \left|\mathcal{I}^-(d) - \mathcal{E}^-(d)\right| \le 16d \sup_{0<\delta\le d}\left|\frac{h(1/\delta)}{\delta}\right| = o(d) \tag{5.22}$$

We now conclude our proof putting together the results so far. Consider the expression $\left|\int_{\widetilde{J}} u'^2 - l'^2\right| + |\mathcal{I}^+(d) - \mathcal{E}^+(d) - \mathcal{I}^-(d) + \mathcal{E}^-(d)|$. In (5.19), (5.22) we have made an estimate from above for this quantity independent of the particular competitor u but dependent only on the data of the problem such as the properties of w and h. So we can choose $\eta(d) > 0$ such that

$$o(d) = \left|\int_{\widetilde{J}} u'^2 - l'^2\right| + |\mathcal{I}^+(d) - \mathcal{E}^+(d) - \mathcal{I}^-(d) + \mathcal{E}^-(d)| \leq \eta(d)$$

with $\eta(d)$ independent of u depending only on d. Using this η choose $\varphi^{(2)}$ as specified (5.13). The arguments above allow us to suppose that η was chosen to be continuous and so $\varphi^{(2)}$ will also be such.

We write using the positivity of $\varphi^{(1)}$ that and (5.21)

$$\begin{aligned}\mathscr{F}(u) - \mathscr{F}(w) = \int_I \left(\varphi(x, u - w) + u'^2 - w'^2\right) \geq \\ \geq \int_{\widetilde{J}} \left(\varphi^{(2)}(x, u - w) + u'^2 - w'^2\right) + \int_{I \setminus \widetilde{J}} \left(\varphi^{(1)}(x, u - w) + u'^2 - w'^2\right) \geq \\ \geq -|\mathcal{I}^+(d) - \mathcal{E}^+(d) - \mathcal{I}^-(d) + \mathcal{E}^-(d)| - \left|\int_{\widetilde{J}} u'^2 - l'^2\right| + \int_{\widetilde{J}} \varphi^{(2)}(x, u - w)\end{aligned}$$

By our choice of $\varphi^{(2)}$ and by (5.14), since $J \subset \widetilde{J}$, this concludes the proof that w is a minimizer since we have

$$\mathscr{F}(u) - \mathscr{F}(w) > 0$$

$\square$

5.4 Examples and relation to Tonelli's partial regularity theorem

Let's now look at the two examples 5.3 and 5.7 in the light of the results of section 4 and of theorem 4.1. It is easily verified that all hypothesis except (4.1) are satisfied. Let's consider the behavior of C_R (the Lipschitz constant in z) in more detail.

Remark 5.8. *For examples 5.3 and 5.7, (4.1) holds only if $C_R(x) > |w''(x)|$ a.e. in x. We have that $w''(x)$ is not integrable around the singularity point $x = 0$. So the hypothesis of theorem 4.1 are not satisfied.*

Proof. The only part of the Lagrangian $F(x, z, p)$ of the form (5.1), used in the examples, that depend on z are the penalty functions. Let's restrict our attention to these terms when evaluating the Lipschitz continuity in z. Consider the points $(x, w(x), 0)$ and $(x, w(x) + \Delta, 0)$ with $0 < |\Delta| < |x|$. For example 5.3 we have that

$$F(x, w(x) + \Delta, 0) - F((x, w(x), 0)) = \varphi(x, w(x) + \Delta, 0) - \varphi(x, w(x), 0) = |\Delta||w''(x)|$$

This implies that for any $R > 0$ we must have $C_R(x) > |w''(x)|$ a.e.. Now we prove that $|w''(x)|$ is not integrable around $x = 0$, due to the oscillatory nature of w. The conditions on the Dini derivatives in (5.2) guarantee the existence of $x_n, \widetilde{x}_n \to 0$ such that

$$\frac{w(x_n)}{x_n} = 1 + o_n(1)$$
$$\frac{w(\widetilde{x}_n)}{\widetilde{x}_n} = -1 + o_n(1)$$

and the sequences can be chosen alternating so that $0 < x_{n+1} < \widetilde{x}_n < x_n$. By the Lagrange theorem for each n there is a sequence of points $\theta_n \in (x_{n+1}, \widetilde{x}_n)$ such that $w'(\theta_n) = +1 + o_n(1)$ and a sequence $\widetilde{\theta}_n \in (\widetilde{x}_n, x_n)$ such that $w'(\widetilde{\theta}_n) = -1 + o_n(1)$. This implies that $\int_{\theta_n}^{\widetilde{\theta}} |w''| \geq |\int_{\theta_n}^{\widetilde{\theta}} w''| = 2 + o_n$. Then we have

$$\int_I C_R > \sum_{n=0}^{\infty} \int_{\theta_n}^{\widetilde{\theta}} |w''| = +\infty.$$

Notice that this result actually shows that the choice of w so that $xw''(x) = o(1)$ from (5.2) holds is rather delicate since $|w''(x)|$ cannot be integrable if it oscillates. For example 5.7 essentially the same estimates can be done to the right and to the left of 0 noticing that $\varphi(x, w(x), 0) = 0$ and $\varphi(x, w(x) + \Delta, 0) \geq \varphi^{(1)}(x, w(x) + \Delta, 0)$. $\square$

5.5 Further examples of minimizers with many singularities

It is clear that the example 5.7 can be easily extended by replicating the construction for any finite number of singularity points in $\overline{I}$. However a legitimate doubt can be that the number of these singularities where Tonelli regularity fails is in some sense small. In other words one can suppose that Tonelli regularity holds if one eliminates from $\overline{I}$ some small closed set Σ. However Gratwick and Preiss in their paper show that this is not the case. They generalize the examples shown in this paper by constructing a functional that has a minimizer with singularities, similar to the one in 0 of w, on the image of any initially fixed sequence. This is done by successively inserting small local copies of w around 0 for each point x_n of the sequence. The penalty function is constructed as a series accordingly. The base ideas are same as in example 5.7 but there are lots of technicalities necessary to guarantee the convergence for the Lagrangian. The interested reader is invited to consult [6] while were we limit ourselves to stating the result and noticing some interesting consequences.

Example 5.9 (Gratwick and Preiss's example for a functional with continuous Lagrangian and singular AC minimizer)**.** *Let $I = (-T, T)$ be some interval and let $\Sigma \subset I$ be an arbitrary numerable set. Then there exists a continuous Lagrangian $F : \overline{I} \times \mathbb{R} \times \mathbb{R} \to [0, +\infty)$, quadratic in p with $F_{pp} > 0$ for all $(x, z, p) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$ such that the associated functional $\mathscr{F}(u) = \int_I F(x, u, u')$ has a minimizer $\widetilde{w}$ among $\mathcal{C} = \{u \in AC([-T, T]) \mid u = w \text{ on } \partial I\}$ such that Tonelli regularity for $\widetilde{w}$ fails at least on Σ. $\widetilde{w}$ is Lipschitz continuous and for $x \in \Sigma$, $\overline{D}\widetilde{w}(x) \geq 1$ and $\underline{D}\widetilde{w}(x) \leq -1$.*

Corollary 5.10. *Actually, using the previous example, the set where Tonelli regularity fails can be greater than the sole numerable set Σ. Set Σ to be dense, for example the*

$\Sigma := \mathbb{Q} \cap I$. *As a matter of fact setting*

$$\Sigma_k^{\pm} = \left\{ t \in \overline{I} \mid \left| \frac{\widetilde{w}(s) - \widetilde{w}(t)}{s-t} - \pm 1 \right| < \frac{1}{k} \text{ for some } s \in \overline{I} \cap [t - \frac{1}{k}, t + \frac{1}{k}] \right\}$$

we have that

$$\widetilde{\Sigma} = \{ t \in \overline{I} \mid \overline{D}\widetilde{w}(t) \geq 1 \text{ and } \underline{D}\widetilde{w}(t) \leq -1 \} = \bigcap_{k \in \mathbb{N}} \Sigma_k^+ \cap \Sigma_k^-$$

is a dense G_δ (second category) set.

This example clears any reasonable doubts about the possibility of eliminating the singularities of $\widetilde{w}$.

6 Appendix

Here we state some useful results for one-dimensional calculus of variations in AC. For a good introduction to these methods see [2].

First we state that for the class of Lagrangians one naturally deals with, the associated functionals are well defined.

Theorem 6.1 (Measurability criterion). *Let*

$$\begin{array}{rccl} f & : & \mathbb{R}^N \times \mathbb{R}^K & \longrightarrow & \mathbb{R} \\ & & (x,p) & \longmapsto & f(x,p) \end{array}$$

be measurable in x for all p and continuous in p for almost all x. Then, for any measurable $\xi : \mathbb{R}^N \to \mathbb{R}^K$, $f(x, \xi(x))$ is measurable.

We state a theorem due to Tonelli that guarantees the existence of minimizers for functionals among AC functions.

Theorem 6.2 (Tonelli's existence theorem). *Let $\mathscr{F}$ be a functional with Lagrangian $F : \overline{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying these conditions:*

1. *$F(x,z,p)$ is continuous.*
2. *$F(x,z,p)$ is convex in p.*
3. *$F(x,z,p)$ has super-linear growth i.e. there exists $\theta : (0; +\infty) \to \mathbb{R}$ convex and strictly increasing, such that*

$$\lim_{t \to +\infty} \frac{\theta(t)}{t} = +\infty$$
$$F(x,z,p) > \theta(p) \quad \forall (x,z,p) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$$

then $\mathscr{F}$ has a minimizer in the class

$$\mathcal{C} = \{ u \in AC(I) \quad u = u_0 \text{ on } \partial I \}.$$

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